

THE INTERSECTION OF A CURVE WITH A UNION OF TRANSLATED CODIMENSION 2 SUBGROUPS IN A POWER OF AN ELLIPTIC CURVE

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ABSTRACT. Let E be an elliptic curve. Consider an irreducible algebraic curve C embedded in E^g . The curve is transverse if it is not contained in any translate of a proper algebraic subgroup of E^g . Furthermore C is weak-transverse if it is not contained in any proper algebraic subgroup. Suppose that both E and C are defined over the algebraic numbers.

We prove that the algebraic points of a transverse curve C which are close to the union of all algebraic subgroups of E^g of codimension 2 translated by points in a subgroup Γ of E^g of finite rank are a set of bounded height. The notion of close is defined using a height function. If Γ is trivial, it is sufficient to suppose that C is weak-transverse.

Then, we introduce a method to determine the finiteness of these sets. From a conjectural lower bound for the normalised height of a transverse curve C , we deduce that the above sets are finite. At present, such a lower bound exists for $g \leq 3$.

Our results are optimal, for what concerns the codimension of the algebraic subgroups.

1. INTRODUCTION

We present the problems in a general context.

Denote by A a semi-abelian variety over $\overline{\mathbb{Q}}$ of dimension g . Consider an irreducible algebraic subvariety V of A , defined over $\overline{\mathbb{Q}}$. We say that

- V is transverse if V is not contained in any translate of a proper algebraic subgroup of A .
- V is weak-transverse if V is not contained in any proper algebraic subgroup of A .

Given an integer r with $1 \leq r \leq g$ and a subset F of $A(\overline{\mathbb{Q}})$, we define the set

$$S_r(V, F) = V(\overline{\mathbb{Q}}) \cap \left(\bigcup_{\text{cod } B \geq r} B + F \right),$$

where B varies over all semi-abelian subvarieties of A of codimension at least r and

$$B + F = \{b + f : b \in B, f \in F\}.$$

For $r > g$, we define $S_r(V, F)$ to be the empty set. We denote the set $S_r(V, A_{\text{Tor}})$ simply by $S_r(V)$. Note that

$$S_{r+1}(V, F) \subset S_r(V, F).$$

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A natural question to ask would be; for which sets F and integers r , the set $S_r(V, F)$ is non Zariski-dense in V .

Sets of this kind, for $r = g$, appear in the literature in the context of the Mordell-Lang, of the Manin-Mumford and of the Bogomolov Conjectures. More recently Bombieri, Masser and Zannier [3] have proven that the set $S_2(C)$ is finite, for a transverse curve C in a torus. They investigate, for the first time, intersections with the union of all algebraic subgroups of a given codimension. This opens a vast number of conjectures for subvarieties of semi-abelian varieties.

In this article we consider the elliptic case for curves. Let E be an elliptic curve and C an irreducible algebraic curve in E^g , both defined over $\overline{\mathbb{Q}}$. Let $\|\cdot\|$ be a semi-norm on $E^g(\overline{\mathbb{Q}})$ induced by a height function. For $\varepsilon \geq 0$, we denote

$$\mathcal{O}_\varepsilon = \{\xi \in E^g(\overline{\mathbb{Q}}) : \|\xi\| \leq \varepsilon\}.$$

Let $\Gamma \subseteq E^g(\overline{\mathbb{Q}})$ be a subgroup of finite rank. Define $\Gamma_\varepsilon = \Gamma + \mathcal{O}_\varepsilon$.

Conjecture 1.1. *Let $C \subset E^g$. Then,*

- i. *If C is weak-transverse, $S_2(C)$ is finite.*
- ii. *If C is transverse, $S_2(C, \Gamma)$ is finite.*
- iii. *If C is weak-transverse, there exists $\varepsilon > 0$ such that $S_2(C, \mathcal{O}_\varepsilon)$ is finite.*
- iv. *If C is transverse, there exists $\varepsilon > 0$ such that $S_2(C, \Gamma_\varepsilon)$ is finite.*

The strong hypotheses of C transverse or the weak hypotheses of C weak-transverse is a crucial difference in the setting. Please take note as to which hypotheses is assumed in the different statements.

Clearly iv. implies ii. by setting $\varepsilon = 0$, and similarly iii. implies i.

The union of all algebraic subgroups of codimension g is exactly the torsion of E^g . Then, $C \cap \Gamma_\varepsilon \subset S_g(C, \Gamma_\varepsilon) \subset S_2(C, \Gamma_\varepsilon)$. So, Conjecture 1.1 iii. and iv. imply the Bogomolov ([15], [17]) and the Mordell-Lang plus Bogomolov ([11]) Theorems respectively.

Partial results related to i. and ii. have been proven. In [16] we solve a weak form of i. There we assume the stronger hypothesis that C is transverse. If E has C.M. then $S_2(C)$ is finite. If E has no C.M. then $S_{\frac{g}{2}+2}(C)$ is finite. In [12] Rémond and the author present a weak version of ii. Again if E has C.M. the result is optimal. If E has no C.M. the codimension of the algebraic subgroups depends on Γ . In addition, we show that i. and ii. are equivalent. Note that there are no trivial implications between iii. and iv., because of the different hypotheses on C .

These known proofs rely on Northcott's theorem: a set is finite if and only if it has bounded height and degree. To prove that the degree is bounded one uses Siegel's Lemma and an essentially optimal generalized Lehmer's Conjecture. Up to a logarithm factor, the generalized Lehmer conjecture is presently known for a point in a torus [1] and in a C.M. abelian variety [4]. This method has some disadvantages: first it is only known to work for transverse curves and for $\varepsilon = 0$, secondly a quasi optimal generalized Lehmer's Conjecture is not likely to be proven in a near future for a general abelian variety.

In this article we introduce a different method. First, we bound the height also for weak-transverse curves.

Theorem 1.2. *There exists $\varepsilon > 0$ such that:*

- i. *If C is weak-transverse, $S_2(C, \mathcal{O}_\varepsilon)$ has bounded height.*
- ii. *If C is transverse, $S_2(C, \Gamma_\varepsilon)$ has bounded height.*

The proof of both statements uses a Vojta inequality as stated in [12] Proposition 2.1. The second assertion is proven in [12] Theorem 1.5. To prove the first assertion (see section 7), we embed $S_2(C, \mathcal{O}_\varepsilon)$ into two sets associated to a transverse curve. We then manage to apply a Vojta inequality on each of these two sets.

As a second result, we prove:

Theorem 1.3. *For $r \geq 2$, the following statements are equivalent:*

- i. *If C is weak-transverse, then there exists $\varepsilon > 0$ such that $S_r(C, \mathcal{O}_\varepsilon)$ is finite.*
- ii. *If C is transverse, then there exists $\varepsilon > 0$ such that $S_r(C, \Gamma_\varepsilon)$ is finite.*

This theorem is not as easy as the equivalence of Conjecture 1.1 part i. and ii. That i. implies ii. is quite elementary. The other implication is delicate. In particular we make use of Theorem 1.2 (see section 7).

In the third instance, we show how to avoid the use of the Siegel Lemma and of the generalized Lehmer Conjecture. Instead, we use Dirichlet's Theorem and a conjectural effective version of the Bogomolov Theorem. Bogomolov's Theorem states that the set of points of small height on a curve of genus at least 2 is finite. We define $\mu(C)$ as the supremum of the reals $\epsilon(C)$ such that $S_g(C, \mathcal{O}_{\epsilon(C)}) = C \cap \mathcal{O}_{\epsilon(C)}$ is finite. The essential minimum of C is $\mu(C)^2$ (note that in the literature, often, the notation \mathcal{O}_ε corresponds to the set, we denote in this work, $\mathcal{O}_{\varepsilon^2}$; thus in the references given below the bounds are given for the essential minimum and not for its square root $\mu(C)$ as we use here).

Non-optimal effective lower bounds for $\mu(C)$ are given by S. David and P. Philippon [6] Theorem 1.4 and [7] Theorem 1.6. The lower bound we need is the elliptic analogue to a theorem of Amoroso and David. In [2] Theorem 1.4, they prove an essentially optimal lower bound for a variety in a torus. The following conjecture is a weak form of [7] Conjecture 1.5 part ii. where the line bundle is fixed.

Conjecture 1.4. *Let $A = E_1 \times \cdots \times E_g$ be a product of elliptic curves defined over a number field k . Let L be the tensor product of the pull-backs of symmetric line bundles on E_i via the natural projections. Let $C \subset A$ be an irreducible transverse curve defined over $\overline{\mathbb{Q}}$. Let η be any positive real. Then, there exists a constant $c(g, A, \eta) = c(g, \deg_L A, h_L(A), [k : \mathbb{Q}], \eta)$ such that, for*

$$\epsilon(C, \eta) = c(g, A, \eta)(\deg_L C)^{-\frac{1}{2(g-1)} - \eta},$$

the set

$$C(\overline{\mathbb{Q}}) \cap \mathcal{O}_{\epsilon(C, \eta)}$$

is finite.

In section 11, we prove:

Theorem 1.5. *Conjecture 1.4 implies Conjecture 1.1.*

Conjecture 1.4 can be stated for subvarieties of A . Galateau [8] proves that such a conjecture holds for varieties of codimension 1 or 2 in a product of elliptic curves. Then, for $g \leq 3$, Conjecture 1.1 holds unconditionally.

Theorems 1.2 and 1.5 are optimal with respect to the codimension of the algebraic subgroups - see remark 9.2.

We have already pointed out that Conjecture 1.1 implies the Bogomolov Conjecture and the Mordell-Lang plus Bogomolov Theorem. Let us emphasise that our Theorem 1.5 does not give a new proof of the Bogomolov Conjecture, as we assume such an effective result. On the other hand, it gives a new proof of the Mordell-Lang plus Bogomolov Theorem, under the assumption of Conjecture 1.4.

The proof of Theorem 1.5 is based on two steps. A union of infinitely many sets is finite if and only if

- (1) the union can be taken over finitely many sets,
- (2) all sets in the union are finite.

(1) is a typical problem of Diophantine approximation. The proof relies on Dirichlet's Theorem on the rational approximation of reals. The fact that we consider

small neighbourhoods enables us to move the algebraic subgroups ‘a bit’. So we can consider only subgroups of bounded degree, which are finitely many (see Proposition A, §12).

The second step (2) places itself in the context of the height theory. Its proof relies on Conjecture 1.4. The bound $\epsilon(C, \eta)$ depends on the invariants of the ambient variety and on the degree of C . A weaker dependence on the degree of C would not be enough for our application. Also the non-dependence of the bound on the field of definition of C proves useful. Playing on Conjecture 1.4, we produce a sharp lower bound for the essential minimum of the image of a curve under certain morphisms (see Proposition B, §13).

The effectiveness aspect of our method is noteworthy; the use of a Vojta inequality makes Theorem 1.2, and consequently Theorem 1.5, ineffective. Though, the rest of the method is effective. Indeed, in section 14, we prove a weaker, but effective analogue of Theorem 1.5.

Theorem 1.6. *Assume Conjecture 1.4. Let C be transverse. Then, there exists an effective $\varepsilon > 0$ such that the set $S_2(C, \mathcal{O}_\varepsilon)$ is finite.*

A bound for the number of points of small height on the curve would then imply a bound for the cardinality of $S_2(C, \mathcal{O}_\varepsilon)$ for C transverse and ε small (see Theorem 14.3).

The toric version of Theorem 1.6 is independently studied by P. Habegger in his Ph.D. thesis [9]. He follows the idea of using a Bogomolov type bound, proven in the toric case in [2] Theorem 1.4. He proves the finiteness of $S_2(C, \mathcal{O}_\epsilon)$, for $\epsilon > 0$ and C a transverse curve in a torus.

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2. PRELIMINARIES

2.1. Morphisms and their height. Let $(R, |\cdot|)$ be a hermitian ring, that means R is a domain and $|\cdot|$ an absolute value on R .

We denote by $M_{r,g}(R)$ the module of $r \times g$ matrices with entries in R .

For $F = (f_{ij}) \in M_{r,g}(R)$, we define the height of F as the maximum of the absolute value of its entries

$$H(F) = \max_{ij} |f_{ij}|.$$

Let E be an elliptic curve defined over a number field. The ring of endomorphism $\text{End}(E)$ is isomorphic either to \mathbb{Z} (if E does not have C.M.) or to an order in an imaginary quadratic field (if E has C.M.). We consider on $\text{End}(E)$ the standard absolute value of \mathbb{C} . Note that this absolute value does not depend on the embedding of $\text{End}(E)$ in \mathbb{C} .

We identify a morphism $\phi : E^g \rightarrow E^r$ with a matrix in $M_{r,g}(\text{End}(E))$. Note that the set of morphism of height bounded by a constant is finite.

An intrinsic definition of absolute value on $\text{End}(E)$ can be given using the Rosati-involution.

In the following, we aim to be as transparent as possible, polishing statements from technicality. Therefore, we principally present proofs for E without C.M. Then $\text{End}(E)$ is identified with \mathbb{Z} and a morphism ϕ with an integral matrix. In the final section, we explain how to deal with the technical complication of a ring of endomorphisms of rank 2 and with a product of elliptic curves instead of a power.

2.2. Small points. On E , we fix a symmetric very ample line bundle \mathcal{L} . On E^g , we consider the bundle L which is the tensor product of the pull-backs of \mathcal{L} via the natural projections on the factors. Degrees are computed with respect to the polarization L .

Usually $E^g(\overline{\mathbb{Q}})$ is endowed with the L -canonical Néron-Tate height h' . Though, to simplify constants, we prefer to define on E^g the height of the maximum

$$h(x_1, \dots, x_g) = \max_i (h(x_i)).$$

where $h(\cdot)$ on $E(\overline{\mathbb{Q}})$ is the \mathcal{L} -canonical Néron-Tate height. The height h is the square of a norm $\|\cdot\|$ on $E^g(\overline{\mathbb{Q}}) \otimes \mathbb{R}$. For a point $x \in E^g(\overline{\mathbb{Q}})$, we write $\|x\|$ for $\|x \otimes 1\|$.

Note that $h(x) \leq h'(x) \leq gh(x)$. Hence, the two norms induced by h and h' are equivalent.

For $a \in \text{End}(E)$, we denote by $[a]$ the multiplication by a . For $y \in E^g(\overline{\mathbb{Q}})$ it holds

$$\|[a]y\| = |a| \cdot \|y\|.$$

The height of a non-empty set $S \subset E^g(\overline{\mathbb{Q}})$ is the supremum of the heights of its elements. The norm of S is the non-negative square root of its height.

For $\varepsilon \geq 0$, we denote

$$\mathcal{O}_\varepsilon = \mathcal{O}_{\varepsilon, E^g} = \{\xi \in E^g(\overline{\mathbb{Q}}) : \|\xi\| \leq \varepsilon\}.$$

2.3. Subgroups. Let M be a R -module. The R -rank of M is the supremum of the cardinality of a set of R -linearly independent elements of M . If M has finite rank s , a maximal free set of M is a set of s linearly independent elements of M . If M is a free R -module of rank s , we call a set of s generators of M , integral generators of M .

Please note that a free \mathbb{Z} -module of finite rank is a lattice; in the literature, what we call integral generators can be called basis, and what we define as maximal free

set is a basis of the vector space given by tensor product with the quotient field of R .

We say that $(M, \|\cdot\|)$ is a hermitian R -module if M is an R -module and $\|\cdot\|$ is a norm on the tensor product of M with the quotient field of R . For an element $p \in M$ we write $\|p\|$ for $\|p \otimes 1\|$.

Let E be an elliptic curve. In the following, we will simply say module for an $\text{End}(E)$ -module.

Note that any subgroup of $E^g(\overline{\mathbb{Q}})$ of finite rank is contained in a sub-module of finite rank. Conversely, a sub-module of E^g of finite rank is a subgroup of finite rank.

Let Γ be a subgroup of finite rank of $E^g(\overline{\mathbb{Q}})$. We define

$$\Gamma_\varepsilon = \Gamma + \mathcal{O}_\varepsilon.$$

The saturated module Γ_0 of the coordinates group of Γ (in short of Γ) is a sub-module of $E(\overline{\mathbb{Q}})$ defined as

$$(1) \quad \Gamma_0 = \{\phi(y) \in E \text{ for } \phi : E^g \rightarrow E \text{ and } Ny \in \Gamma \text{ with } N \in \mathbb{Z}^*\}.$$

Note that $\Gamma_0^g = \Gamma_0 \times \cdots \times \Gamma_0$ is a sub-module of E^g invariant via the image or preimage of isogenies. Furthermore, it contains Γ and it is a module of finite rank. This shows that to prove finiteness statements for Γ it is enough to prove them for Γ_0^g .

We denote by s the rank of Γ_0 . Let $\gamma_1, \dots, \gamma_s$ be a maximal free set of Γ_0 . We denote the associated point of E^s by

$$\gamma = (\gamma_1, \dots, \gamma_s).$$

3. SOME GEOMETRY OF NUMBERS

We present a property of geometry of numbers, we then extend it to points of $E^g(\overline{\mathbb{Q}})$. The idea is that, if in \mathbb{R}^n we consider n linearly independent vectors, and we move them within a ‘small’ angle, they will still be linearly independent. Furthermore, the norm of a linear combination of such vectors depends on the norm of these vectors, on their angles, and on the norm of the coefficients of the combination.

Such estimates are frequent in the geometry of numbers.

The following lemma is a reformulation of [13] Theorem 1.1 of Schlickewei or [16] Lemma 3.

Lemma 3.1. *Every hermitian free \mathbb{Z} -module of rank n admits integral generators ρ_1, \dots, ρ_n such that for all integers α_i*

$$c_0(n) \sum_i |\alpha_i|^2 \|\rho_i\|^2 \leq \left\| \sum_i \alpha_i \rho_i \right\|^2$$

with $c_0(n)$ a constant depending only on n .

Proof. A hermitian free \mathbb{Z} -module $(\Gamma, \|\cdot\|)$ of rank n is a lattice in the metric space $\Gamma_{\mathbb{R}}$ given by tensor product with \mathbb{R} . The proof is now equal to the proof of [16] Lemma 3 page 57, from line 19 onwards, where one shall read n for r and ρ_i for g_i . \square

This lemma allows us to explicit the comparison constant for two norms on a finite dimensional vector space over the quotient field of R .

Proposition 3.2. *Let $(M, \|\cdot\|)$ be a hermitian R -module, with R a finitely generated free \mathbb{Z} -module. Let p_1, \dots, p_s be R -linearly independent elements of M . Then there exists an effective positive constant $c_1(p, \tau)$ such that, for all $b_1, \dots, b_s \in R$, it holds*

$$c_1(p, \tau) \sum_i |b_i|_R^2 \|p_i\|^2 \leq \left\| \sum_i b_i p_i \right\|^2,$$

where $p = (p_1, \dots, p_s)$ and $\tau = (1, \tau_2, \dots, \tau_t)$ integral generators of R .

Proof. The sub-module of M defined by $\Gamma_{\mathbb{Z}} = \langle p_1, \dots, p_s, \dots, \tau_t p_1, \dots, \tau_t p_s \rangle_{\mathbb{Z}}$ has rank st over \mathbb{Z} . Clearly, for $1 \leq i \leq t$ and $1 \leq j \leq s$ the elements $\tau_i p_j$ are integral generators of $\Gamma_{\mathbb{Z}}$. Consider the normed space $(M \otimes_{\mathbb{Z}} \mathbb{R}, \|\cdot\|)$, in which $\Gamma_{\mathbb{Z}}$ is embedded, and endow $\Gamma_{\mathbb{Z}}$ with the induced metric.

Apply Lemma 3.1 to $(\Gamma_{\mathbb{Z}}, \|\cdot\|)$ with $n = st$. Then, there exist integral generators ρ_1, \dots, ρ_{st} of $\Gamma_{\mathbb{Z}}$ satisfying

$$(2) \quad \begin{aligned} \left\| \sum_i \alpha_i \rho_i \right\|^2 &\geq c_0(st) \sum_i |\alpha_i|^2 \|\rho_i\|^2 \\ &\geq c_0(st) \sum_i |\alpha_i|^2 \min_k \|\rho_k\|^2, \end{aligned}$$

for all $\alpha_1, \dots, \alpha_{st} \in \mathbb{Z}$.

We decompose the elements $b_1, \dots, b_s \in R$ as

$$b_i = \sum_{j=1}^t \alpha_{ij} \tau_j$$

with $\alpha_{ij} \in \mathbb{Z}$. We denote

$$\alpha = (\alpha_{11}, \dots, \alpha_{1t}, \dots, \alpha_{s1}, \dots, \alpha_{st}) \in \mathbb{Z}^{st}.$$

As usually $(\cdot)^t$ indicates the transpose, we denote

$$p^\tau = (\tau_1 p_1, \dots, \tau_t p_1, \tau_1 p_2, \dots, \tau_t p_2, \dots, \tau_1 p_s, \dots, \tau_t p_s)^t \in \Gamma_{\mathbb{Z}}^{st}$$

and

$$\rho = (\rho_1, \dots, \rho_{st})^t \in \Gamma_{\mathbb{Z}}^{st}.$$

Let $P \in SL_{st}(\mathbb{Z})$ be the base change matrix such that

$$p^\tau = P\rho.$$

Then

$$\sum_i b_i p_i = \sum_{ij} \alpha_{ij} \tau_j p_i = \alpha \cdot p^\tau = \alpha \cdot (P\rho) = (\alpha P) \cdot \rho.$$

Passing to the norms and using relation (2) with the coefficients $(\alpha_1, \dots, \alpha_{st}) = \alpha P$, we deduce

$$\left\| \sum_i b_i p_i \right\|^2 = \|(\alpha P) \cdot \rho\|^2 \geq c_0(st) |\alpha P|_2^2 \min_k \|\rho_k\|^2,$$

where $|\cdot|_2$ is the standard Euclidean norm. On the other hand, the triangle inequality gives

$$\begin{aligned} |b_i|_R^2 &\leq \max_k |\tau_k|_R^2 \left(\sum_{j=1}^t |\alpha_{ij}| \right)^2 \\ &\leq t \max_k |\tau_k|_R^2 \sum_{j=1}^t |\alpha_{ij}|^2. \end{aligned}$$

We deduce

$$\frac{\|\sum_i b_i p_i\|^2}{\sum_i |b_i|_R^2 \|p_i\|^2} \geq \frac{c_0(st)}{t \max_j |\tau_j|_R^2} \frac{\min_i \|\rho_i\|^2}{\max_i \|p_i\|^2} \frac{|\alpha P|_2^2}{|\alpha|_2^2}.$$

We shall still estimate $\frac{|\alpha P|_2^2}{|\alpha|_2^2}$ independently of α . For a linear operator A and a row vector β , it holds the classical norm relation $|\beta A|_2 \leq H(A) |\beta|_2$. For $A = P^{-1}$ and $\beta = \alpha P$, we deduce

$$\frac{|\alpha P|_2^2}{|\alpha|_2^2} \geq \frac{1}{H(P^{-1})^2}.$$

Then

$$\frac{\|\sum_i b_i p_i\|^2}{\sum_i |b_i|_R^2 \|p_i\|^2} \geq \frac{c_0(st)}{t \max_j |\tau_j|_R^2} \frac{\min_i \|\rho_i\|^2}{\max_i \|p_i\|^2} \frac{1}{H(P^{-1})^2}$$

or equivalently

$$\left\| \sum b_i p_i \right\|^2 \geq c_1(p, \tau) \sum_i |b_i|_R^2 \|p_i\|^2,$$

where

$$(3) \quad c_1(p, \tau) = \frac{c_0(st)}{t \max_j |\tau_j|_R^2} \frac{\min_i \|\rho_i\|^2}{\max_i \|p_i\|^2} \frac{1}{H(P^{-1})^2}.$$

□

The following non surprising proposition has some surprising implications; it allows us to prove Theorems 1.2 and 1.3.

Proposition 3.3. *Let p_1, \dots, p_s be linearly independent points of $E(\overline{\mathbb{Q}})$ and $p = (p_1, \dots, p_s)$. Let τ be a set of integral generators of $\text{End}(E)$. Then, there exist positive reals $c_2(p, \tau)$ and $\varepsilon_0(p, \tau)$ such that*

$$c_2(p, \tau) \sum_i |b_i|^2 \|p_i\|^2 \leq \left\| \sum_i b_i (p_i - \xi_i) - b\zeta \right\|^2$$

for all $b_1, \dots, b_s, b \in \text{End}(E)$ with $|b| \leq \max_i |b_i|$ and for all $\xi_1, \dots, \xi_s, \zeta \in E(\overline{\mathbb{Q}})$ with $\|\xi_i\|, \|\zeta\| \leq \varepsilon_0(p, \tau)$.

In particular $p_1 - \xi_1, \dots, p_s - \xi_s$ are linearly independent points of E .

Proof. Recall that the norm on $\text{End}(E)$ is compatible with the height norm on $E(\overline{\mathbb{Q}})$. Namely $\|b_i p_i\| = |b_i|_{\text{End}(E)} \|p_i\|$. Thus $(\text{End}(E), |\cdot|)$ is a hermitian free \mathbb{Z} -module of rank 1 if E has not C.M. or 2 is E has C.M. Furthermore, $(E, \|\cdot\|)$ is a hermitian $\text{End}(E)$ -module.

Apply Proposition 3.2 with $R = \text{End}(E)$, $M = E$ and $\tau = (1)$ if $\text{End}(E) \cong \mathbb{Z}$ or $\tau = (1, \tau_2)$ if $\text{End}(E) \cong \mathbb{Z} + \tau_2 \mathbb{Z}$. We deduce that, for $b_1, \dots, b_s \in \text{End}(E)$

$$(4) \quad \left\| \sum b_i p_i \right\|^2 \geq c_1(p, \tau) \sum_i |b_i|^2 \|p_i\|^2.$$

Let $\|\xi_i\|, \|\zeta\| \leq \varepsilon$. Since $|b| \leq \max |b_i|$ the triangle inequality implies

$$\begin{aligned} \left\| \sum_i b_i (p_i - \xi_i) - b\zeta \right\| &\geq \left\| \sum_i b_i p_i \right\| - \varepsilon \sum_i |b_i| - \varepsilon |b| \\ &\geq \left\| \sum_i b_i p_i \right\| - 2\varepsilon \sum_i |b_i|. \end{aligned}$$

Pass to the squares and do not forget that $(\sum_{i=1}^s |b_i|)^2 \leq s \sum_{i=1}^s |b_i|^2$. We deduce

$$\begin{aligned} \left\| \sum_i b_i(p_i - \xi_i) - b\zeta \right\|^2 &\geq \left\| \sum_i b_i p_i \right\|^2 - 4\varepsilon \left\| \sum_i b_i p_i \right\| \sum_i |b_i| + 4\varepsilon^2 \left(\sum_i |b_i| \right)^2 \\ &\geq \left\| \sum_i b_i p_i \right\|^2 - 4s\varepsilon \left(\sum_i |b_i|^2 \right) \max_i \|p_i\|. \end{aligned}$$

Choose

$$(5) \quad \varepsilon \leq \varepsilon_0(p, \tau) = \frac{c_1(p, \tau)}{8s} \frac{\min_i \|p_i\|^2}{\max_i \|p_i\|}.$$

Using relation (4), we deduce

$$\begin{aligned} \left\| \sum_i b_i(p_i - \xi_i) - b\zeta \right\|^2 &\geq c_1(p, \tau) \sum_i |b_i|^2 \|p_i\|^2 - \frac{1}{2} c_1(p, \tau) \left(\sum_i |b_i|^2 \right) \min_i \|p_i\|^2 \\ &\geq \frac{1}{2} c_1(p, \tau) \sum_i |b_i|^2 \|p_i\|^2. \end{aligned}$$

Set for example

$$(6) \quad c_2(p, \tau) = \frac{1}{2} c_1(p, \tau),$$

where $c_1(p, \tau)$ is as defined in relation (3).

In particular such a relation, with $b = 0$, implies that only the trivial linear combination of $p_1 - \xi_1, \dots, p_s - \xi_s$ is zero. \square

At last, we write a lemma which enables us to choose a nice maximal free set of Γ_0 , the saturated module of a sub-module Γ of $E(\overline{\mathbb{Q}})$ of finite rank, as defined in relation (1). There is nothing deep here, as we are working on finite dimensional \mathbb{C} -vector spaces.

Lemma 3.4 (Quasi orthonormality). *Let Γ_0 be the saturated module of Γ . Let s be the rank of Γ_0 . Then for any real $K > 0$, there exists a maximal free set $\gamma_1, \dots, \gamma_s$ of Γ_0 such that $\|\gamma_i\| \geq K$ and for all $b_1, \dots, b_s \in \text{End}(E)$*

$$\left\| \sum_i b_i \gamma_i \right\|^2 \geq \frac{1}{9} \sum_i |b_i|^2 \|\gamma_i\|^2.$$

Proof. Recall that $\text{End}(E)$ is an order in an imaginary quadratic field k . Furthermore, the height norm $\|\cdot\|$ makes Γ_0 a hermitian $\text{End}(E)$ -module. Let Γ^{free} be a submodule of Γ_0 isomorphic to its free part. Then Γ^{free} is a k vector space of dimension s . Its tensor product with \mathbb{C} over k is a normed \mathbb{C} vector space of dimension s , and Γ^{free} is isomorphic to $\Gamma^{\text{free}} \otimes 1$. Using for instance the Gram-Schmidt orthonormalisation algorithm in $\Gamma^{\text{free}} \otimes_k \mathbb{C}$, we can choose an orthonormal basis

$$v_i = g_i \otimes \rho_i.$$

So

$$\left\| \sum_i b_i v_i \right\|^2 = \sum_i |b_i|^2.$$

Decompose $\rho_i = r_{i1} + \tau r_{i2}$ for $1, \tau$ integral generators of $\text{End}(E)$ and $r_{ij} \in \mathbb{R}$. Choose $\delta = (2(1 + |\tau|) \max_i \|g_i\|)^{-1}$ and rationals q_{ij} such that $q_{ij} = r_{ij} + d_{ij}$ with $|d_{ij}| \leq \delta$ (use the density of the rationals).

Define

$$\gamma'_i = g_i \otimes (q_{i1} + \tau q_{i2}) = (q_{i1} + \tau q_{i2}) g_1 \otimes 1 \in \Gamma^{\text{free}} \otimes 1,$$

and

$$\delta_i = g_i \otimes (d_{i1} + \tau d_{i2}).$$

Then

$$v_i = \gamma'_i + \delta_i$$

with

$$\|\delta_i\| \leq \|g_i\|(1 + |\tau|)\delta \leq \frac{1}{2}.$$

The triangle inequality gives

$$2\left\|\sum_i b_i \gamma'_i\right\|^2 \geq \left\|\sum_i b_i v_i\right\|^2 - 2\left\|\sum_i b_i \delta_i\right\|^2.$$

The orthonormality of v_i and $\|\delta_i\| \leq \frac{1}{2}$ imply that

$$\begin{aligned} 2\left\|\sum_i b_i \gamma'_i\right\|^2 &\geq \sum_i |b_i|^2 - 2 \sum_i |b_i|^2 \frac{1}{4} \\ &= \frac{1}{2} \sum_i |b_i|^2. \end{aligned}$$

Finally $\|\gamma'_i\| \leq \|v_i\| + \|\delta_i\| \leq \frac{3}{2}$, so

$$\left\|\sum_i b_i \gamma'_i\right\|^2 \geq \frac{1}{9} \sum_i |b_i|^2 \|\gamma'_i\|^2.$$

It is evident that for any integer n_0 the same relation holds

$$\left\|\sum_i b_i n_0 \gamma'_i\right\|^2 \geq \frac{1}{9} \sum_i |b_i|^2 \|n_0 \gamma'_i\|^2.$$

Let n_0 be an integer such that $n_0 \geq 2K$. Note that $\|\gamma'_i\| \geq \|v_i\| - \|\delta_i\| \geq \frac{1}{2}$, so

$$\|n_0 \gamma'_i\| \geq K.$$

We conclude that the maximal free set $\gamma_i = n_0 \gamma'_i$ satisfies the desired conditions. \square

Remark that we cannot directly choose an orthonormal basis in Γ^{free} , because the norm has values in \mathbb{R} and not in \mathbb{Q} . Actually, we could prove that for any small positive real δ , there exists a maximal free set $\gamma_1, \dots, \gamma_s$ such that

$$\left\|\sum_i b_i \gamma_i\right\|^2 \geq \frac{(1 - \delta)^2}{(1 + \delta)^2} \sum_i |b_i|^2 \|\gamma_i\|^2.$$

We also observe that we could use Proposition 3.2 for any maximal free set $\gamma_1, \dots, \gamma_s$ of Γ_0 , and carry around the related constant $c_1(\gamma, \tau)$. However, we prefer absolute constants, when possible.

4. GAUSS-REDUCED MORPHISMS

The aim of this section is to show that we can consider our union over Gauss-reduced algebraic subgroups, instead of the union over all algebraic subgroups.

Let B be an algebraic subgroup of E^g of codimension r . Then $B \subset \ker \phi_B$ for a surjective morphism $\phi_B : E^g \rightarrow E^r$. Conversely, we denote by B_ϕ the kernel of a surjection $\phi : E^g \rightarrow E^r$. Then B_ϕ is an algebraic subgroup of E^g of codimension r .

The matrices in $M_{r \times g}(\text{End}(E))$ of the form

$$\phi = (aI_r | L) = \begin{pmatrix} a & \dots & 0 & a_{1,r+1} & \dots & a_{1,g} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & a & a_{r,r+1} & \dots & a_{r,g} \end{pmatrix},$$

with $H(\phi) = |a|$ and no common factors of the entries (up to units), will play a key role in this work. For $r = g$, such a morphism becomes the identity, and L shall be forgotten. These matrices have three main advantages:

- i. The restriction of ϕ to the set $E^r \times \{0\}^{g-r}$ is nothing else than the multiplication by a .
- ii. The image of $\mathcal{O}_\varepsilon \subset E^g$ under ϕ is contained in the image of $\mathcal{O}_{g\varepsilon} \cap E^r \times \{0\}^{g-r}$. Similarly, the image of Γ_0^g under ϕ is contained in the image of $\Gamma_0^r \times \{0\}^{g-r}$.
- iii. The matrix ϕ has small height compared to other matrices with same zero component of the kernel.

Definition 4.1 (Gauss-reduced Morphisms). *We say that a surjective morphism $\phi : E^g \rightarrow E^r$ is Gauss-reduced of rank r if:*

- i. *There exists $a \in \text{End}(E)^*$ such that aI_r is a submatrix of ϕ , with I_r the r -identity matrix,*
- ii. *$H(\phi) = |a|$,*
- iii. *If there exists $f \in \text{End}(E)$ and $\phi' : E^g \rightarrow E^r$ such that $\phi = f\phi'$ then f is an isomorphism.*

We say that an algebraic subgroup is Gauss-reduced if it is the kernel of a Gauss-reduced morphism.

Remark 4.2. *Note that if $\text{End}(E) \cong \mathbb{Z}$ the condition iii. in this definition simply says that the greatest common divisor of the entries of ϕ is 1 and $f = \pm 1$.*

Whenever we will ensure $\text{End}(E) \cong \mathbb{Z}$, we will require, in the definition of Gauss-reduced 4.1 ii., the more restrictive condition $H(\phi) = a$, instead of $H(\phi) = |a|$. Obviously $B_\phi = B_{-\phi}$, thus all lemmas below hold with this ‘up to units’-definition of Gauss-reduced. This assumption simplifies notations.

A morphisms ϕ' , given by a reordering of the rows of a morphism ϕ , have the same kernel as ϕ . Saying that aI_r is a sub-matrix of ϕ fixes one permutation of the rows of ϕ .

A reordering of the columns corresponds, instead, to a permutation of the co-ordinates. Statements will be proven for Gauss-reduced morphisms of the form $\phi = (aI|L)$. For each other reordering of the columns the proofs are analogous. Since there are finitely many permutations of g columns, the finiteness statements will follow.

The following lemma is a simple useful trick to keep in mind.

Lemma 4.3. *Let $\phi : E^g \rightarrow E^r$ be Gauss-reduced of rank r .*

- i. *For $\xi = (\xi_1, \dots, \xi_g) \in \mathcal{O}_\varepsilon$, there exists a point $\xi' = (\xi'', \{0\}^{g-r}) \in \mathcal{O}_{g\varepsilon}$ such that*

$$\phi(\xi) = \phi(\xi') = [a]\xi''.$$

- ii. *For $y = (y_1, \dots, y_g) \in \Gamma_0^g$, there exists a point $y' = (y'', \{0\}^{g-r}) \in \Gamma_0^r \times \{0\}^{g-r}$ such that*

$$\phi(y) = \phi(y') = [a]y''.$$

Proof. Up to reordering of the columns, the morphism ϕ has the form

$$\phi = \begin{pmatrix} a & \dots & 0 & a_{1,r+1} & \dots & a_{1,g} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & a & a_{r,r+1} & \dots & a_{r,g} \end{pmatrix},$$

with $H(\phi) = |a|$.

- i. Consider a point $\xi'' \in E^r$ such that

$$[a]\xi'' = \phi(\xi).$$

Since $\|\xi''\| = \frac{\|\phi(\xi)\|}{|a|} = \max_i \frac{\|\sum_j a_{ij}\xi_j\|}{|a|}$ and $|a| = \max_{ij} |a_{ij}|$, we obtain

$$\|\xi''\| \leq g\varepsilon.$$

Define $\xi' = (\xi'', \{0\}^{g-r})$. Clearly

$$\phi(\xi') = [a]\xi'' = \phi(\xi).$$

ii. Note that, $\phi(y) \in \Gamma_0^r$. Since Γ_0 is a division group, the point y'' such that

$$[a]y'' = \phi(y),$$

belongs to Γ_0^r . Define $y' = (y'', \{0\}^{g-r})$. Then

$$\phi(y') = [a]y'' = \phi(y).$$

□

In the following lemma, we show that the zero components of B_ϕ , for ϕ ranging over all Gauss-reduced morphisms of rank r , are all possible abelian subvarieties of E^g of codimension r . This is proven using the classical Gauss algorithm, where the pivots have maximal absolute values.

Lemma 4.4. *Let $\psi : E^g \rightarrow E^r$ be a morphism of rank r . Then*

i. *For every $N \in \text{End}(E)^*$, it holds*

$$B_{N\psi} \subset B_\psi + (E_{\text{Tor}}^r \times \{0\}^{g-r}).$$

ii. *There exists a Gauss-reduced morphism $\phi : E^g \rightarrow E^r$ of rank r such that*

$$B_\psi \subset B_\phi + (E_{\text{Tor}}^r \times \{0\}^{g-r}).$$

Proof. i. We show the first relation.

Let $b \in B_{N\psi}$, then $N\psi(b) = 0$. So $\psi(b) = t$ with t a N -torsion point in E^r . Let ψ_1 be an invertible r -submatrix of ψ . Up to reordering of the columns, we can suppose $\psi = (\psi_1 | \psi_2)$. Let t' be a torsion point in E^r such that $\psi_1(t') = t$. Then $\psi(b - (t', 0)) = 0$. Thus $b \in B_\psi + (E_{\text{Tor}}^r \times \{0\}^{g-r})$.

ii. We show the second relation.

The Gauss algorithm gives an invertible integral r -matrix Δ such that, up to the order of the columns, $\Delta\psi$ is of the form

$$\Delta\psi = \begin{pmatrix} a & \dots & 0 & a_{1,r+1} & \dots & a_{1,g} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & a & a_{r,r+1} & \dots & a_{r,g} \end{pmatrix},$$

with $H(\Delta\psi) = |a|$ (potentially there are common factors of the entries).

Let $b \in B_\psi$, then $\psi(b) = 0$. Hence $\Delta\psi(b) = 0$. It follows

$$B_\psi \subset B_{\Delta\psi}.$$

Let $N \in \text{End}(E)^*$ such that $N|\Delta\psi$ and such that if $f|(\Delta\psi/N)$ then f is a unit (if $\text{End}(E) \cong \mathbb{Z}$, then N is simply the greatest common divisor of the entries of $\Delta\psi$). We define

$$\phi = \Delta\psi/N.$$

Clearly ϕ is Gauss-reduced and $B_\psi \subset B_{\Delta\psi} = B_{N\phi}$. By part i. of this lemma applied to $N\phi$, we conclude

$$B_\psi \subset B_\phi + (E_{\text{Tor}}^r \times \{0\}^{g-r}).$$

□

Note that, in the previous lemma, a reordering of the columns of ψ or ϕ induces the same reordering of the coordinates of $E_{\text{Tor}}^r \times \{0\}^{g-r}$.

Taking intersections with the algebraic points of our curve, the previous lemma part ii. translates immediately as

Lemma 4.5. *Let $C \subset E^g$ be an algebraic curve (transverse or not). For any real $\varepsilon \geq 0$*

$$S_r(C, (\Gamma_0^g)_\varepsilon) = \bigcup_{\substack{\phi \text{ Gauss-reduced} \\ \text{rk}(\phi)=r}} C(\overline{\mathbb{Q}}) \cap (B_\phi + (\Gamma_0^g)_\varepsilon).$$

Proof. By definition

$$S_r(C, (\Gamma_0^g)_\varepsilon) \supseteq \bigcup_{\substack{\phi \text{ Gauss-reduced} \\ \text{rk}(\phi)=r}} C(\overline{\mathbb{Q}}) \cap (B_\psi + (\Gamma_0^g)_\varepsilon).$$

On the other hand, by the previous Lemma 4.4 ii, we see that

$$C(\overline{\mathbb{Q}}) \cap (B_\psi + (\Gamma_0^g)_\varepsilon) \subset C(\overline{\mathbb{Q}}) \cap (B_\phi + (E_{\text{Tor}}^r \times \{0\}^{g-r}) + (\Gamma_0^g)_\varepsilon),$$

with ϕ Gauss-reduced of rank r . Moreover $(E_{\text{Tor}}^r \times \{0\}^{g-r}) \subset \mathcal{O}_\varepsilon \subset (\Gamma_0^g)_\varepsilon$. \square

5. RELATION BETWEEN TRANSVERSE AND WEAK-TRANSVERSE CURVES

We discuss here how we can associate to a couple (C, Γ) , with C a transverse curve and Γ a subgroup of finite rank, a weak-transverse curve C' and vice versa. There are properties which are easier for C and others for C' . Using this association, we will try to gain advantages from both situations.

5.1. From transverse to weak-transverse. Let C be transverse in E^g . If Γ has rank 0, we set $C' = C$. If $\text{rk } \Gamma \geq 1$, consider the saturated module Γ_0 of rank s associated to Γ , as defined in relation (1). Let $\gamma_1, \dots, \gamma_s$ be a maximal free set of Γ_0 . We denote the associated point of E^s by

$$\gamma = (\gamma_1, \dots, \gamma_s).$$

We define

$$C' = C \times \gamma.$$

Since C is transverse and the γ_i are $\text{End}(E)$ -linearly independent, the curve C' is weak-transverse. More precisely, suppose on the contrary that C' would be contained in an algebraic subgroup B_ϕ of codimension 1, with $\phi = (a_1, \dots, a_{g+s})$. Define y_1 to be a point in E such that $a_1 y_1 = \sum_{i=g+1}^{g+s} a_i \gamma_{i-g}$ and define $y = (y_1, 0, \dots, 0) \in E^g$. Then $C \subset B_{\phi_1} + y$ with $\phi_1 = (a_1, \dots, a_g)$, contradicting that C is transverse.

5.2. From weak-transverse to transverse. Let C' be weak-transverse in E^n . If C' is transverse, we set $C = C'$ and $\Gamma = 0$. Suppose that C' is not transverse. Let H_0 be the abelian subvariety of smallest dimension g such that $C' \subset H_0 + p$ for $p \in H_0^\perp(\overline{\mathbb{Q}})$ and H_0^\perp the orthogonal complement of H_0 with respect to the canonical polarization.

Then E^n is isogenous to $H_0 \times H_0^\perp$. Furthermore H_0 is isogenous to E^g and H_0^\perp is isogenous to E^s where $s = n - g$. Let j_0, j_1 and j_2 be such isogenies. We fix the isogeny

$$j = (j_1 \times j_2) \circ j_0 : E^n \rightarrow H_0 \times H_0^\perp \rightarrow E^g \times E^s,$$

which sends H_0 to $E^g \times 0$ and H_0^\perp to $0 \times E^s$.

Then

$$j(C') \subset (E^g \times 0) + j(p),$$

with $j(p) = (0, \dots, 0, p_1, \dots, p_s)$.

We consider the natural projection on the first g coordinates

$$\begin{aligned}\pi : E^g \times E^s &\rightarrow E^g \\ j(C') &\rightarrow \pi(j(C')).\end{aligned}$$

We define

$$C = \pi(j(C')) \quad \text{and} \quad \Gamma = \langle p_1, \dots, p_s \rangle^g.$$

Since H_0 has minimal dimension, the curve C is transverse in E^g .

Note that

$$j(C') = C \times (p_1, \dots, p_s).$$

In addition $j(C')$ is weak-transverse, because C' is. Therefore, $\langle p_1, \dots, p_s \rangle$ has rank s ; indeed if $\sum_{i=1}^s a_i p_i = 0$, then $j(C') \subset B_\phi$ for $\phi = (\{0\}^g, a_1, \dots, a_s)$.

5.3. Weak-transverse up to an isogeny. Statements on boundedness of heights or finiteness of sets are invariant under an isogeny of the ambient variety. Namely, given an isogeny j of E^g , Theorems 1.2 and 1.5 hold for a curve if and only if they hold for its image via j . Thus, the previous discussion shows that without loss of generality, we can assume that a weak-transverse curve C' in E^n is of the form

$$C' = C \times p$$

with

- i. C transverse in E^g ,
- ii. $p = (p_1, \dots, p_s)$ a point in E^s such that the module $\langle p_1, \dots, p_s \rangle$ has rank s ,
- iii. $n = g + s$.

This simplifies the setting for weak-transverse curves.

5.4. Implying the Mordell-Lang plus Bogomolov Theorem for curves.

Note that

$$S_g(C, \mathcal{O}_\varepsilon) = C \cap \mathcal{O}_\varepsilon$$

and

$$S_g(C(\Gamma_0^g)_\varepsilon) = C \cap (\Gamma_0^g)_\varepsilon.$$

Moreover $S_2(C, \cdot) \supset S_g(C, \cdot)$. This immediately shows that Conjecture 1.1 implies the Bogomolov Theorem for weak-transverse curves and the Mordell-Lang plus Bogomolov Theorem for transverse curves. We want to show that Conjecture 1.1 implies these theorems for all curves of genus ≥ 2 .

In E^g a curve of genus 2 is a translate of an elliptic curve isogenous to E . If C is not transverse, then $C \subsetneq H_0 + p$ with H_0 an algebraic subgroup of minimal dimension satisfying such inclusion. Let $\pi : E^g \rightarrow E^g/H_0^\perp$ be the natural projection and let $\psi : E^g/H_0^\perp \rightarrow E^k$ be an isogeny. Then $\|\psi\pi(x)\| \ll \|x\|$. In E^k , consider the transverse curve $C' = \psi\pi(C - p)$ and $\Gamma' = \psi\pi\langle \Gamma, \Gamma_p \rangle$. Note that $\psi\pi(\text{Tor}_{E^g}) \subset \text{Tor}_{E^k}$. Then

$$S_g(C, (\Gamma_0^g)_\varepsilon) \subset \pi_{|C}^{-1} S_k(C', (\Gamma_0'^g)_\varepsilon).$$

The map $\pi_{|C}^{-1}$ has finite fiber. Applying Conjecture 1.1 to $C' \subset E^k$ we deduce that $S_g(C, (\Gamma_0^g)_\varepsilon)$ is finite.

Note that such a proof works only for $S_g(C, \cdot)$. Indeed the projection $\psi\pi(B) \subset E^k$ of an algebraic subgroup B of E^g of codimension r , might not be of codimension r in E^k . Eventually, it could be the entire E^k .

6. QUASI-SPECIAL MORPHISMS

As Gauss-reduced morphisms play a key role for transverse curves, quasi-special morphisms play a key role for weak-transverse curves. In particular, for small ε , quasi-special morphisms will be enough to cover the whole $S_r(C \times p, \mathcal{O}_\varepsilon)$ - see Lemma 6.2 below.

Let us give a flavor for quasi-special. Suppose that $C \times p$ is weak-transverse in E^{g+s} with C transverse in E^g . A point of $C \times p$ is of the form (x, p) . The last s -coordinates are constant and just the x varies. This two parts must be treated differently. Saying that a morphism $\tilde{\psi} = (\psi|\psi')$ is quasi-special ensures that the rank of ψ is maximal (note that ψ acts on x). In particular, this allows us to apply the Gauss algorithm on the first g columns of $\tilde{\psi}$.

Definition 6.1 (Quasi-special). *A surjective morphism $\tilde{\phi} : E^{g+s} \rightarrow E^r$ is quasi-special if there exist $N \in \text{End}(E)^*$, morphisms $\phi : E^g \rightarrow E^r$ and $\phi' : E^s \rightarrow E^r$ such that*

- i. $\tilde{\phi} = (N\phi|\phi')$,
- ii. $\phi = (aI_r|L)$ is Gauss-reduced of rank r ,
- iii. If there exists $f \in \text{End}(E)$ and $\tilde{\phi}' : E^{g+s} \rightarrow E^r$ such that $\tilde{\phi} = f\tilde{\phi}'$ then f is an isomorphism.

Note that we do not require that $\tilde{\phi}$ is Gauss-reduced, the fact is that $H(\phi')$ might not be controlled by $NH(\phi)$. This extra condition will define special morphisms (see Definition 10.1).

Lemma 6.2. *Let $C \times p$ be weak-transverse in E^{g+s} with C transverse in E^g . Then, there exists $\varepsilon > 0$ such that*

$$S_r(C \times p, \mathcal{O}_\varepsilon) \subset \bigcup_{\substack{\tilde{\phi} \text{ quasi-special} \\ \text{rk } \tilde{\phi} = r}} (C(\overline{\mathbb{Q}}) \times p) \cap (B_{\tilde{\phi}} + \mathcal{O}_\varepsilon).$$

We can choose $\varepsilon \leq \varepsilon_0(p, \tau)$, where $\varepsilon_0(p, \tau)$ is as in Proposition 3.3.

Proof. Let $(x, p) \in S_r(C \times p, \mathcal{O}_\varepsilon)$. Then $(x, p) \in (C(\overline{\mathbb{Q}}) \times p) \cap (B_{\tilde{\psi}} + \mathcal{O}_\varepsilon)$ for a morphism $\tilde{\psi} = (\psi|\psi') : E^{g+s} \rightarrow E^r$ of rank r . In other words, there exists a point $(\xi, \xi') \in \mathcal{O}_\varepsilon$ such that

$$\tilde{\psi}((x, p) + (\xi, \xi')) = 0.$$

First, we show that the rank of ψ is r . Suppose, on the contrary, that the rank of ψ would be less than r . Then a linear combination of the rows of ψ is trivial, namely

$$(\lambda_1, \dots, \lambda_r)\psi = 0.$$

Since $\psi(x + \xi) + \psi'(p + \xi') = 0$, the same linear combination of the r coordinates of $\psi'(p + \xi')$ is trivial, namely

$$(\lambda_1, \dots, \lambda_r)\psi'(p + \xi') = 0.$$

Apply Proposition 3.3 with $(b_1, \dots, b_s) = (\lambda_1, \dots, \lambda_r)\psi'$, $(\xi_1, \dots, \xi_s) = -\xi'$, $\zeta = 0$ and $b = 0$. This implies that, if $\varepsilon \leq \varepsilon_0(p, \tau)$, then the points $p_1 + \xi'_1, \dots, p_s + \xi'_s$ are linearly independent. It follows that

$$(\lambda_1, \dots, \lambda_r)\psi' = 0.$$

Hence, the rank of $\tilde{\psi}$ would be less than r , contradicting the fact that the rank of $\tilde{\psi}$ is r .

Since the rank of ψ is r , we can apply the Gauss algorithm using pivots in ψ of maximal absolute values in ψ (clearly we cannot require that they have maximal

absolute values in $\tilde{\psi}$). Let Δ be an invertible matrix, given by the Gauss algorithm, such that $\Delta\tilde{\psi} = (\phi_1|\phi_2)$ with fI_r a submatrix of ϕ_1 .

We shall still get rid of possible common factors. Let $N_1, n_1 \in \text{End}(E)^*$ such that $N_1|\phi_1$ and $n_1|\Delta\tilde{\psi}$. Further suppose that, if $f|(\phi_1/N_1)$ or $f|(\Delta\tilde{\psi}/n_1)$ then f is a unit of $\text{End}(E)$ (if $\text{End}(E) \cong \mathbb{Z}$, then N_1 is the greatest common divisor of the entries of ϕ_1 and n_1 the greatest common divisor of the entries of $\Delta\tilde{\psi}$). Then

$$\Delta\tilde{\psi} = n_1(N\phi|\phi')$$

with $N = N_1/n_1$, $\phi = \phi_1/N_1$ and $\phi' = \phi_2/n_1$. We define

$$\tilde{\phi} = (N\phi|\phi').$$

Clearly $\tilde{\phi}$ is quasi-special. In addition

$$B_{\tilde{\psi}} \subset B_{\Delta\tilde{\psi}} = B_{n_1\tilde{\phi}}.$$

By Lemma 4.4 i. (with $\psi = \tilde{\phi}$ and $N = n_1$) we deduce that

$$B_{\tilde{\psi}} \subset B_{\tilde{\phi}} + E_{\text{Tor}}^r \times \{0\}^{g+s-r}.$$

Since $(x, p) \in B_{\tilde{\psi}} + \mathcal{O}_\varepsilon$, then $(x, p) \in B_{\tilde{\phi}} + \mathcal{O}_\varepsilon$ with $\tilde{\phi}$ quasi-special. \square

7. ESTIMATES FOR THE HEIGHT: THE PROOF OF THEOREM 1.2

As it has been already pointed out, Theorem 1.2 part ii. is proven in [12] Theorem 1.5. In this section, we adapt the proof of [12] Theorem 1.5 to Theorem 1.2 part i.

In view of section 5.3, we can assume, without loss of generality, that a weak-transverse curve C' in E^n has the form

$$C' = C \times p$$

with

- i. C transverse in E^g ,
- ii. $p = (p_1, \dots, p_s)$ a point in E^s such that the module $\langle p_1, \dots, p_s \rangle$ has rank s ,
- iii. $n = g + s$.

Definition 7.1. Let p be a point in E^s and ε a non negative real. We define G_p^ε as the set of points $\theta \in E^2$ for which there exist a matrix $A \in M_{2,s}(\text{End}(E))$, an element $a \in \text{End}(E)$ with $0 < |a| \leq H(A)$, points $\xi \in E^s$ and $\zeta \in E^2$ of norm at most ε such that

$$[a]\theta = A(p + \xi) + [a]\zeta.$$

We identify G_p^ε with the subset $G_p^\varepsilon \times \{0\}^{g-2}$ of E^g .

Now we embed $S_2(C \times p, \mathcal{O}_\varepsilon)$ in two sets related to the transverse curve C . We then use the Vojta inequality on these new sets.

Lemma 7.2. The natural projection on the first g coordinates

$$\begin{aligned} E^g \times E^s &\rightarrow E^g \\ (x, y) &\rightarrow x \end{aligned}$$

defines an injection

$$S_2(C \times p, \mathcal{O}_{\varepsilon/2gs}) \hookrightarrow S_2(C, (\Gamma_p^g)_\varepsilon) \cup \bigcup_{\substack{\phi: E^g \rightarrow E^2 \\ \text{Gauss-reduced}}} C(\overline{\mathbb{Q}}) \cap B_\phi + G_p^\varepsilon.$$

Proof. Let $(x, p) \in S_2(C \times p, \mathcal{O}_{\varepsilon/2gs})$. By Lemma 6.2, $(x, p) \in B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/2gs}$, with $\tilde{\phi} = (N\phi|\phi') : E^{g+s} \rightarrow E^2$ quasi-special of rank 2. Hence

$$\tilde{\phi}((x, p) + (\xi, \xi')) = 0,$$

for $(\xi, \xi') \in \mathcal{O}_{\varepsilon/2gs}$. We can write the equality as

$$N\phi(x) + N\phi(\xi) + \phi'(p + \xi') = 0.$$

By definition of quasi-special ϕ is Gauss-reduced, so

$$\phi = (aI_2|L).$$

By Lemma 4.3 i applied to ϕ and ξ , we can assume

$$\xi = (\xi_1, \xi_2, 0, \dots, 0) \in \mathcal{O}_{\varepsilon/2s}.$$

Suppose first that $NH(\phi) \geq H(\tilde{\phi})$. Let ζ be a point in $E^2 \times \{0\}^{g-2}$ such that

$$N[a]\zeta = (\phi'(\xi'), 0, \dots, 0).$$

Then

$$\|\zeta\| = \frac{\|\phi'(\xi')\|}{NH(\phi)} \leq \frac{\varepsilon}{2}.$$

Let y be a point in $E^2 \times \{0\}^{g-2}$ such that

$$N[a]y = (\phi'(p), 0, \dots, 0).$$

Since Γ_p is a division group, $y \in \Gamma_p^2 \times \{0\}^{g-2}$. Then

$$N\phi(x + \xi + \zeta + y) = 0$$

with $y + \xi + \zeta \in \Gamma_p^g + \mathcal{O}_\varepsilon$. So

$$x \in S_2(C, (\Gamma_p^g)_\varepsilon).$$

Suppose now that $NH(\phi) < H(\tilde{\phi})$ or equivalently $NH(\phi) < H(\phi')$. Let θ' be a point in E^2 such that

$$N[a]\theta' = \phi'(p + \xi') + N[a]\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

and $\theta = (\theta', \{0\}^{g-r})$. Then $\theta \in G_p^\varepsilon$. Moreover

$$\begin{aligned} N\phi(x + \theta) &= N\phi(x) + N\phi(\theta) \\ &= N\phi(x) + N[a]\theta' \\ &= N\phi(x) + \phi'(p + \xi') + N[a]\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \\ &= N\phi(x) + N\phi(\xi) + \phi'(p + \xi') \\ &= \tilde{\phi}((x, p) + (\xi, \xi')) = 0. \end{aligned}$$

Thus

$$x \in B_{N\phi} + G_p^\varepsilon,$$

and by Lemma 4.4 i.

$$x \in B_\phi + (E_{\text{Tor}}^2 \times \{0\}^{g-2}) + G_p^\varepsilon.$$

Note that $G_p^\varepsilon + (E_{\text{Tor}}^2 \times \{0\}^{g-2}) \subset G_p^\varepsilon$. Hence,

$$x \in C(\overline{\mathbb{Q}}) \cap B_\phi + G_p^\varepsilon.$$

□

Lemma 7.3 (Equivalent of [12] Lemma 3.2). *For $\phi : E^g \rightarrow E^2$ Gauss-reduced of rank 2, we have the following inclusion of sets*

$$(B_\phi + G_p^\varepsilon) \subset \{P + \theta : P \in B_\phi, \theta \in G_p^\varepsilon \text{ and } \max(\|\theta\|, \|P\|) \leq 2g\|P + \theta\|\}.$$

Proof. Let $x \in (B_\phi + G_p^\varepsilon)$ with $\phi = (aI_r|L)$ Gauss reduced of rank 2. Then $x = P + \theta$ with $P \in B_\phi$ and $\theta \in G_p^\varepsilon$ and $\phi(x - \theta) = 0$. By definition $G_p^\varepsilon \subset E^2 \times \{0\}^{g-2}$, so $\phi(\theta) = [a]\theta$. Then

$$\|\theta\| = \frac{\|\phi(\theta)\|}{H(\phi)} = \frac{\|\phi(x)\|}{H(\phi)} \leq g\|x\|.$$

So

$$\|P\| = \|x - \theta\| \leq (g + 1)\|x\| = (g + 1)\|P + \theta\|.$$

□

Note that, [12] Lemma 3.3 part (1) is a statement on the morphism, therefore it holds with no need of any remarks.

Lemma 7.4 (Equivalent of [12] Lemma 3.3 part (2)). *There exists an effective $\varepsilon_2 > 0$ such that, for all $\varepsilon \leq \varepsilon_2$, any sequence of elements in G_p^ε admits a sub-sequence in which every two elements θ, θ' satisfy*

$$\left\| \frac{\theta}{\|\theta\|} - \frac{\theta'}{\|\theta'\|} \right\| \leq \frac{1}{16gc_1},$$

where c_1 depends on C and is as defined in [12] Proposition 2.1.

Proof. We decompose two elements θ and θ' in a given sequence of elements of G_p^ε as follows

$$[a]\theta = A(p + \xi) + [a]\zeta$$

and

$$[a']\theta' = A'(p + \xi') + a'\zeta'$$

with $A, A' \in M_{2,s}(\text{End}(E))$ and $0 < |a| \leq H(A)$, $0 < |a'| \leq H(A')$. Let us define y and y' such that

$$[a]y = A(p)$$

and

$$[a']y' = A'(p).$$

Since the sphere of radius 1 is compact in

$$(\langle p_1, \dots, p_s \rangle \times \langle p_1, \dots, p_s \rangle) \otimes \mathbb{R},$$

we can extract a sub-sequence such that for every two elements y and y' it holds

$$\left\| \frac{y}{\|y\|} - \frac{y'}{\|y'\|} \right\| \leq \frac{1}{48gc_1}.$$

Note that

$$\left\| \frac{\theta}{\|\theta\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{A(\xi) + [a]\zeta}{A(p + \xi) + [a]\zeta} \right\|$$

and

$$\left\| \frac{y}{\|y\|} - \frac{y'}{\|y'\|} \right\| = \left\| \frac{\|A(p)\| - \|A(p + \xi) + [a]\zeta\|}{\|A(p + \xi) + [a]\zeta\|} \right\| \leq \left\| \frac{A(\xi) + [a]\zeta}{A(p + \xi) + [a]\zeta} \right\|$$

(same relations with $'$). We deduce

$$\begin{aligned} \left\| \frac{\theta}{\|\theta\|} - \frac{\theta'}{\|\theta'\|} \right\| &\leq \left\| \frac{y}{\|y\|} - \frac{y'}{\|y'\|} \right\| + \left\| \frac{y}{\|\theta\|} - \frac{y}{\|y\|} \right\| + \left\| \frac{y'}{\|\theta'\|} - \frac{y'}{\|y'\|} \right\| \\ &\quad + \left\| \frac{\theta}{\|\theta\|} - \frac{y}{\|\theta\|} \right\| + \left\| \frac{\theta'}{\|\theta'\|} - \frac{y'}{\|\theta'\|} \right\| \\ &\leq \left\| \frac{y}{\|y\|} - \frac{y'}{\|y'\|} \right\| + 2 \left\| \frac{A(\xi) + [a]\zeta}{A(p + \xi) + [a]\zeta} \right\| + 2 \left\| \frac{A'(\xi') + [a']\zeta'}{A'(p + \xi') + [a']\zeta'} \right\|. \end{aligned}$$

Choose

$$(7) \quad \varepsilon \leq \varepsilon_2 = \min(\varepsilon_0(p, \tau), \varepsilon'_0(p, \tau)),$$

where $\varepsilon_0(p, \tau)$ is defined in relation (5), $c_2(p, \tau)$ is defined in relation (6) and

$$\varepsilon'_0(p, \tau) = \frac{c_2(p, \tau)^{\frac{1}{2}} \min \|p_i\|}{96(s+1)c_1}.$$

Note that $\|A(p + \xi) + [a]\zeta\| = \|A_k(p + \xi) + a\zeta_k\|$ for $k = 1$ or 2 and $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$. Proposition 3.3 applied with $b_1, \dots, b_s = A_k$, $\xi = -\xi$, $\zeta = -\zeta_k$ and $b = a$, implies

$$\|A(p + \xi) + [a]\zeta\| \geq H(A)c_2(p, \tau)^{\frac{1}{2}} \min \|p_i\|$$

(same relation with $'$).

It follows

$$\begin{aligned} \left\| \frac{\theta}{\|\theta\|} - \frac{\theta'}{\|\theta'\|} \right\| &\leq \frac{1}{48gc_1} + \varepsilon \frac{2H(A)(s+1)}{H(A)c_2(p, \tau)^{\frac{1}{2}} \min \|p_i\|} \\ &\quad + \varepsilon \frac{2H(A')(s+1)}{H(A')c_2(p, \tau)^{\frac{1}{2}} \min \|p_i\|} \\ &\leq \frac{1}{48gc_1} + \frac{1}{48gc_1} + \frac{1}{48gc_1}, \end{aligned}$$

where in the last inequality we use $\varepsilon \leq \varepsilon'_0(p, \tau)$. \square

We are ready to conclude.

Proof of Theorem 1.2 i. In view of Lemma 7.2, we shall prove that there exists $\varepsilon > 0$ such that $S_2(C, (\Gamma_p)_\varepsilon)$ and $\bigcup_{\substack{\phi: E^g \rightarrow E^2 \\ \text{Gauss-reduced}}} C(\overline{\mathbb{Q}}) \cap B_\phi + G_p^\varepsilon$ have bounded height.

By Theorem 1.2 ii., there exists $\varepsilon_1 > 0$ such that for $\varepsilon \leq \varepsilon_1$, the first set has bounded height.

It remains to show that there exists $\varepsilon_2 > 0$ such that, for $\varepsilon \leq \varepsilon_2$, the set

$$\bigcup_{\substack{\phi: E^g \rightarrow E^2 \\ \text{Gauss-reduced}}} C(\overline{\mathbb{Q}}) \cap B_\phi + G_p^\varepsilon$$

has bounded height. The proof follows, step by step, the proof of [12] Theorem 1.5. In view of Lemma 7.3 and 7.4, all conditions for the proof of [12] Theorem 1.5 are satisfied. The proof is then exactly equal to the proof [12] page 1927-1928. \square

Remark 7.5. In [12] Theorem 1.5, we show that for $\varepsilon_1 = \frac{1}{2^g c_1}$ the set $S_2(C, \Gamma_{\varepsilon_1})$ has bounded height. The constant c_1 depends on the invariants of the curve C . This constant is defined in [12] Proposition 2.1 and it is effective. On the other hand, the height of $S_2(C, \Gamma_{\varepsilon_1})$ is bounded by a constant which is not known to be effective, unless Γ has rank 0.

For $C \times p$, we have shown that for $\varepsilon'_2 = \frac{\min(1, c_2(p, \tau)) \min \|p_i\|^2}{2^8 g(s+1)^2 \max \|p_i\| c_1}$ the set $S_2(C \times p, \mathcal{O}_{\varepsilon'_2})$ has bounded height - see relation (7) and Lemma 7.2. As in the previous case, the

height of $S_2(C \times p, \mathcal{O}_{\varepsilon'_2})$ is bounded by a constant which, in general, is not known to be effective.

8. RECAP

We would like to recall and fix the notations for the rest of the article.

For simplicity, we assume that $\text{End}(E) \cong \mathbb{Z}$. In this case the saturated module of a group coincides with its division group. According to Remark 4.2, we use $H(\phi) = a$ in the definition of a Gauss-reduced morphism and $N \in \mathbb{N}^*$ in the definition of quasi-special.

- Let E be an elliptic curve without C.M. over $\overline{\mathbb{Q}}$.
- Let C be a transverse curve in E^g over $\overline{\mathbb{Q}}$.
- Let

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix} = \begin{pmatrix} a & \dots & 0 & L_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & a & L_r \end{pmatrix}$$

be a Gauss-reduced morphism of rank $1 \leq r \leq g$, with $L_i \in \mathbb{Z}^{g-r}$ and $H(\phi) = a$.

- Let Γ be a subgroup of finite rank of $E^g(\overline{\mathbb{Q}})$.
- Let Γ_0 be the division group of Γ and s its rank (the definition is given in relation (1)).
- Choose $\varepsilon_1 > 0$ so that $S_2(C, (\Gamma_0^g)_{\varepsilon_1})$ has bounded height (the definition is consistent in view of Theorem 1.2 ii.).
- Let K_1 be the norm of $S_2(C, (\Gamma_0^g)_{\varepsilon_1})$.
- Let $\gamma = (\gamma_1, \dots, \gamma_s)$ be a point of $E^s(\overline{\mathbb{Q}})$ such that $\gamma_1, \dots, \gamma_s$ is a maximal free set of Γ_0 satisfying the conditions of Lemma 3.4 with $K = 3gK_1$. Namely, for all integers b_i

$$(8) \quad \frac{1}{9} \sum_i |b_i|^2 \|\gamma_i\|^2 \leq \left\| \sum_i b_i \gamma_i \right\|^2$$

and

$$(9) \quad \min_i \|\gamma_i\| \geq 3gK_1.$$

- Let $C \times \gamma$ be the associated weak-transverse curve in E^{g+s} .
- Let $\tilde{\phi} = (N\phi|\phi') : E^{g+s} \rightarrow E^r$ be a quasi-special morphism with $N \in \mathbb{N}^*$.
- Choose $\varepsilon_2 > 0$ so that $S_2(C \times \gamma, \mathcal{O}_{\varepsilon_2})$ has bounded height (the definition is consistent in view of Theorem 1.2 i.).
- Let K_2 be the norm of $S_2(C \times \gamma, \mathcal{O}_{\varepsilon_2})$.
- Let $p = (p_1, \dots, p_s) \in E^s$ be a point such that the rank of $\langle p_1, \dots, p_s \rangle$ is s .
- Let Γ_p be the division group of $\langle p_1, \dots, p_s \rangle$ (in short the division group of p).
- Let c_p and ε_p be the constants $(c_2(p, \tau))^{\frac{1}{2}}$ and $\varepsilon_0(p, \tau)$ defined in Proposition 3.3 for the point p and $\tau = 1$ (please note the square root in c_p).
- Let $C \times p$ be the associated weak-transverse curve in E^{g+s} .
- Choose $\varepsilon_3 > 0$ so that $S_2(C \times p, \mathcal{O}_{\varepsilon_3})$ has bounded height (the definition is consistent in view of Theorem 1.2 i.).
- Let K_3 be the norm of $S_2(C \times p, \mathcal{O}_{\varepsilon_3})$.

9. EQUIVALENCE OF THE STRONG STATEMENTS: THE PROOF OF THEOREM 1.3

The following theorem implies Theorem 1.3 immediately; in addition it gives explicit inclusions. Once more, we would like to emphasise that we need to assume that $S_r(C \times p, \mathcal{O}_\varepsilon)$ has bounded height in order to embed it in a set of the type $S_r(C, \Gamma_{\varepsilon'})$. Therefore we assume $r \geq 2$ and $\varepsilon \leq \varepsilon_3$ in part ii.

Theorem 9.1. *Let $\varepsilon \geq 0$. Then,*

- i. *The map $x \rightarrow (x, \gamma)$ defines an injection*

$$S_r(C, \Gamma_\varepsilon) \hookrightarrow S_r(C \times \gamma, \mathcal{O}_\varepsilon).$$

Recall that γ is a maximal free set of Γ_0 .

- ii. *For $2 \leq r$ and $\varepsilon \leq \min(\varepsilon_p, \varepsilon_3)$, the map $(x, p) \rightarrow x$ defines an injection*

$$S_r(C \times p, \mathcal{O}_\varepsilon) \hookrightarrow S_r(C, (\Gamma_p^g)_\varepsilon K_4),$$

where $K_4 = (g+s) \max\left(1, \frac{g(K_3+\varepsilon)}{c_p \min_i ||p_i||}\right)$. Recall that Γ_p is the division group of p .

Proof. i. Let $x \in S_r(C, \Gamma_\varepsilon)$. Then, there exists a surjective $\phi : E^g \rightarrow E^r$, points $y \in \Gamma$ and $\xi \in \mathcal{O}_\varepsilon$ such that

$$\phi(x + y + \xi) = 0.$$

Since $\gamma = (\gamma_1, \dots, \gamma_s)$ is a maximal free set of Γ_0 , there exists a positive integer N and a matrix $G \in M_{r,s}(\mathbb{Z})$ such that

$$[N]y = G\gamma.$$

We define

$$\tilde{\phi} = (N\phi | \phi G).$$

Then

$$\begin{aligned} \tilde{\phi}((x, \gamma) + (\xi, 0)) &= N\phi(x + \xi) + \phi G(\gamma) \\ &= N\phi(x + \xi + y) = 0. \end{aligned}$$

So

$$(x, \gamma) \in S_r(C \times \gamma, \mathcal{O}_\varepsilon).$$

ii. Let $(x, p) \in S_r(C \times p, \mathcal{O}_\varepsilon)$. Thanks to Lemma 6.2, the assumption $\varepsilon \leq \varepsilon_p$ implies

$$(x, p) \in (B_{\tilde{\phi}} + \mathcal{O}_\varepsilon)$$

with $\tilde{\phi} = (N\phi | \phi')$ quasi-special. Hence

$$\tilde{\phi}((x, p) + (\xi, \xi')) = 0,$$

for $(\xi, \xi') \in \mathcal{O}_\varepsilon$. Equivalently

$$(10) \quad N\phi(x + \xi) = -\phi'(p + \xi').$$

By definition of quasi-special, ϕ is Gauss-reduced of rank r . Let

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix} = \begin{pmatrix} a & \dots & 0 & L_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & a & L_r \end{pmatrix}$$

with $L_i \in \mathbb{Z}^{g-r}$ and $H(\phi) = a$.

Since Γ_p is the division group of p , the point y' defined as

$$N[a]y' = \phi'(p),$$

belongs to Γ_p^r .

Let ζ' be a point of E^r such that

$$N[a]\zeta' = \tilde{\phi}(\xi, \xi').$$

We define

$$\begin{aligned} y &= (y', 0, \dots, 0) \in \Gamma_p^r \times \{0\}^{g-r}, \\ \zeta &= (\zeta', 0, \dots, 0) \in E^r \times \{0\}^{g-r}. \end{aligned}$$

We have

$$\begin{aligned} N\phi(y) &= N[a]y' = \phi'(p) \\ N\phi(\zeta) &= N[a]\zeta' = \tilde{\phi}(\xi, \xi'). \end{aligned}$$

It follows

$$\begin{aligned} N\phi(x + y + \zeta) &= N\phi(x) + \phi'(p) + \tilde{\phi}(\xi, \xi') \\ &= \tilde{\phi}((x, p) + (\xi, \xi')) = 0. \end{aligned}$$

Thus

$$x \in C(\overline{\mathbb{Q}}) \cap (B_{N\phi} + \Gamma_p^g + \mathcal{O}_{\|\zeta\|}).$$

In order to finish the proof, we shall prove

$$\|\zeta\| \leq \varepsilon K_4.$$

By definition of ζ we see that

$$\begin{aligned} \|\zeta\| = \|\zeta'\| &= \frac{\|\tilde{\phi}(\xi, \xi')\|}{Na} \leq (g+s) \frac{\max(H(\phi'), Na)}{Na} \|(\xi, \xi')\| \\ &\leq (g+s) \frac{\max(H(\phi'), Na)}{Na} \varepsilon. \end{aligned}$$

We claim

$$\frac{\max(H(\phi'), Na)}{Na} \leq \frac{K_4}{g+s}.$$

Let $\phi' = (b_{ij})$. We shall prove that $H(\phi') = \max_{ij} |b_{ij}| \leq \frac{K_4}{g+s} Na$. Let $|b_{kl}| = H(\phi')$. Consider the k -row of the system (10)

$$N\phi_k(x) + N\phi_k(\xi) = -\sum_j b_{kj}(p_j + \xi'_j).$$

The triangle inequality gives

$$(11) \quad \frac{\|\phi_k(x)\|}{a} + \frac{\|\phi_k(\xi)\|}{a} \geq \frac{\|\sum_j b_{kj}(p_j + \xi'_j)\|}{Na}.$$

Since $\varepsilon \leq \varepsilon_3$ and $r \geq 2$, then $(x, p) \in S_2(C \times p, \mathcal{O}_{\varepsilon_3})$ which has norm K_3 . Hence

$$\|x\| \leq \|(x, p)\| \leq K_3.$$

Since $a = H(\phi)$, we see that

$$\frac{\|\phi_k(x)\|}{a} \leq (g-r+1)K_3 \quad \text{and} \quad \frac{\|\phi_k(\xi)\|}{a} \leq (g-r+1)\varepsilon.$$

Substituting in (11)

$$(g-r+1)(K_3 + \varepsilon) \geq \frac{\|\sum_j b_{kj}(p_j + \xi'_j)\|}{Na}.$$

Recall that $\varepsilon \leq \varepsilon_p$. Hence, Proposition 3.3 with $(b_1, \dots, b_s) = (b_{k1}, \dots, b_{ks})$ $(\xi_1, \dots, \xi_s) = -\xi'$ and $\zeta = 0$, implies

$$\begin{aligned} (g-r+1)(K_3 + \varepsilon) &\geq \frac{1}{Na} \left(c_p^2 \sum_j |b_{kj}|^2 \|p_j\|^2 \right)^{\frac{1}{2}} \\ &\geq \frac{c_p H(\phi')}{Na} \min_i \|p_i\|. \end{aligned}$$

Whence

$$H(\phi') \leq \frac{K_4}{g+s} Na.$$

□

The inclusion in Theorem 9.1 ii. is proven only for a set $S_r(C \times p, \mathcal{O}_\varepsilon)$ which is known to have bounded height. If the norm K_3 of $S_r(C \times p, \mathcal{O}_\varepsilon)$ goes to infinity, the set $(\Gamma_p^g)_{\varepsilon K_4}$ tends to be the whole E^g .

Remark 9.2. *We would like to show that our Theorems 1.2 and 1.5 are optimal. Let $\Gamma = \langle (y_1, 0, \dots, 0) \rangle$, where y_1 is a non torsion point in $E(\overline{\mathbb{Q}})$. Since C is transverse, the projection π_1 of $C(\overline{\mathbb{Q}})$ on the first factor $E(\overline{\mathbb{Q}})$ is surjective. Let $x_n \in C(\overline{\mathbb{Q}})$ such that $\pi_1(x_n) = ny_1$. So $x_n - n(y_1, 0, \dots, 0)$ has first coordinate zero, and belongs to the algebraic subgroup $0 \times E^{g-1}$. Then, for all $n \in \mathbb{N}$ it holds*

$$x_n \in B_{\phi=(1,0,\dots,0)} + \Gamma.$$

This shows that $x_n \in S_1(C, \Gamma)$, so $S_1(C, \Gamma)$ does not have bounded height. By Theorem 9.1 part i, neither $S_1(C \times y_1)$ has bounded height.

10. SPECIAL MORPHISMS AND AN IMPORTANT INCLUSION

We can actually show a stronger inclusion than the one in Theorem 9.1 i. The set $S_r(C, \Gamma_\varepsilon)$ can be included in a subset of $S_r(C \times \gamma, \mathcal{O}_\varepsilon)$, namely the subset defined by special morphisms.

Definition 10.1 (Special Morphisms). *A surjective morphism $\tilde{\phi} : E^{g+s} \rightarrow E^r$ is special if $\tilde{\phi} = (N\phi|\phi')$ is quasi-special and satisfies the further condition*

$$H(\tilde{\phi}) = NH(\phi).$$

Equivalently $\tilde{\phi}$ is special if and only if

- i. $\tilde{\phi}$ is Gauss-reduced,
- ii. $H(\tilde{\phi})I_r$ is a submatrix of the matrix consisting of the first g columns of $\tilde{\phi}$.

Let us prove the equivalence of these two definitions:

Proof. That the first definition implies the second is a clear matter. For the converse, decompose $\tilde{\phi} = (A|\phi')$ with $A \in M_{r \times g}(\mathbb{Z})$ and $\phi' \in M_{r \times s}(\mathbb{Z})$. Let N be the greatest common divisor of the entries of A . Define $\phi = A/N$ and $a = H(\tilde{\phi})/N$. Then $\phi = (aI_r|L')$ is Gauss-reduced and $\tilde{\phi} = (N\phi|\phi')$. □

A nice remark is that the obstruction to show unconditionally that $S_r(C \times p, \mathcal{O}_\varepsilon)$ is included in $S_r(C, (\Gamma_p^g)_{\varepsilon'})$ is exactly due to the non-special morphisms. Sets of the kind

$$(C(\overline{\mathbb{Q}}) \times p) \cap (B_{\tilde{\phi}} + \mathcal{O}_\varepsilon)$$

which do not have bounded height, can be included in $S_r(C, (\Gamma_p^g)_{\varepsilon'})$ if $\tilde{\phi}$ is special, indeed in general $\varepsilon' = c(g, s) \frac{H(\tilde{\phi})}{H(A)} \varepsilon$ for any $\tilde{\phi} = (A|\phi')$.

Proposition 10.2. *Let $2 \leq r$ and $\varepsilon \leq \min(\varepsilon_1, \frac{K_1}{g})$. The map $x \rightarrow (x, \gamma)$ defines an injection*

$$\bigcup_{\substack{\phi \text{ Gauss reduced} \\ \text{rk}(\phi)=r}} C(\overline{\mathbb{Q}}) \cap (B_\phi + (\Gamma_0^g)_\varepsilon) \hookrightarrow \bigcup_{\substack{\tilde{\phi}=(N\phi|\phi') \text{ special} \\ \text{rk}\tilde{\phi}=r}} (C(\overline{\mathbb{Q}}) \times \gamma) \cap (B_{\tilde{\phi}} + \mathcal{O}_\varepsilon).$$

Proof. Let $x \in C(\overline{\mathbb{Q}}) \cap (B_\phi + \Gamma_0^g + \mathcal{O}_\varepsilon)$ with ϕ Gauss-reduced of rank r . Then, there exist $y \in \Gamma_0^g$ and $\xi \in \mathcal{O}_\varepsilon \subset E^g$ such that

$$\phi(x + y + \xi) = 0.$$

Since $\gamma_1, \dots, \gamma_s$ is a maximal free set of Γ_0 , there exists an integer N and a matrix $G \in M_{r,s}(\mathbb{Z})$ such that

$$[N]y = G(\gamma).$$

Let n be the greatest common divisor of the entries of $(N\phi|\phi G)$. We define

$$\tilde{\phi} = \frac{1}{n}(N\phi|\phi G).$$

Clearly

$$\begin{aligned} (N\phi|\phi G)((x, \gamma) + (\xi, 0)) &= N\phi(x) + \phi G(\gamma) + N\phi(\xi) \\ &= N\phi(x + y + \xi) = 0. \end{aligned}$$

Thus

$$(12) \quad n\tilde{\phi}((x, \gamma) + (\xi, 0)) = 0.$$

Equivalently

$$(x, \gamma) \in (C(\overline{\mathbb{Q}}) \times \gamma) \cap (B_{n\tilde{\phi}} + \mathcal{O}_\varepsilon).$$

By Lemma 4.4 i. with $\psi = \tilde{\phi}$ and $N = n$, it follows

$$(x, \gamma) \in (C(\overline{\mathbb{Q}}) \times \gamma) \cap (B_{\tilde{\phi}} + \mathcal{O}_\varepsilon).$$

We shall still show that $\tilde{\phi}$ is special, using the first definition of special. By assumption, the morphism ϕ is Gauss-reduced. By definition of $\tilde{\phi}$, the greatest common divisor of its entries is 1. In order to conclude that $\tilde{\phi}$ is special, we still have to show that

$$H(\tilde{\phi}) = Na$$

or equivalently

$$H(\phi') \leq Na.$$

The proof is similar to the last part of the proof of Theorem 9.1 ii.

Let $\phi' = (b_{ij}) = \phi G$. Let $|b_{kl}| = \max_{ij} |b_{ij}| = H(\phi')$. Let ϕ_k be the k -th row of ϕ . Consider the k -th row of the system (12)

$$nN(\phi_k(x) + \phi_k(\xi)) = -n \sum_j b_{kj} \gamma_j.$$

Then

$$\frac{\|\phi_k(x)\|}{a} + \frac{\|\phi_k(\xi)\|}{a} \geq \frac{1}{Na} \left\| \sum_j b_{kj} \gamma_j \right\|.$$

Clearly $x \in S_r(C, (\Gamma_0^g)_\varepsilon)$. Since $\varepsilon \leq \varepsilon_1$, then $x \in S_2(C, (\Gamma_0^g)_{\varepsilon_1})$ which has norm bounded by K_1 . So

$$\|x\| \leq K_1.$$

As $H(\phi_k) \leq H(\phi) = a$,

$$\frac{\|\phi_k(x)\|}{a} \leq (g - r + 1)K_1.$$

Furthermore

$$\frac{\|\phi_k(\xi)\|}{a} \leq (g - r + 1)\varepsilon.$$

Then

$$(g - 1)(K_1 + \varepsilon) \geq \frac{1}{Na} \left\| \sum_j b_{kj} \gamma_j \right\|.$$

From relations (8) with $(b_1, \dots, b_s) = (b_{k1}, \dots, b_{ks})$ and (9) in the recap, we deduce

$$\begin{aligned} (g - 1)(K_1 + \varepsilon) &\geq \frac{1}{Na} \left(\frac{1}{9} \sum_j |b_{kj}|^2 \|\gamma_j\|^2 \right)^{\frac{1}{2}} \\ &\geq \frac{H(\phi')}{3Na} \min_j \|\gamma_j\| \\ &\geq \frac{H(\phi')}{3Na} 3gK_1. \end{aligned}$$

We assumed $\varepsilon \leq \frac{K_1}{g}$, so

$$H(\phi') \leq Na.$$

□

This inclusion is important; the Bogomolov type bounds are given for intersections with \mathcal{O}_ε and not with Γ_ε . Actually there exist bounds for ε , such that $C \cap \Gamma_\varepsilon$ is finite. These bounds are deduced by the Bogomolov type bounds and their dependence on the degree of the curve is not sharp enough for our purpose. To overcome such an obstacle and solve the problem with Γ_ε , we use the above Proposition 10.2 and make use of the Bogomolov type bounds for $C \times \gamma$ intersected with $B_{\tilde{\phi}} + \mathcal{O}_\varepsilon$, where $\tilde{\phi}$ is special of rank 2.

11. THE PROOF OF THEOREM 1.5

In sections 12 and 13 below, we prepare the core for the proof of Theorem 1.5. In Proposition A we prove that the union can be taken over finitely many sets, while in Proposition B we prove that each set in the union is finite. Hence, our set is finite.

We prefer to present first the proof of Theorem 1.5 and then to prove the two key Propositions A and B. We hope that, knowing a priori the aim of sections 12 and 13, the reader gets the right inspiration to handle them.

Proof of Theorem 1.5. Assuming Conjecture 1.4, we prove Conjecture 1.1 part iv. In view of Theorem 1.3, also part iii. is proven. Parts i. and ii. are then obtained by setting $\varepsilon = 0$.

Choose

- i. $n = 2(g + s) - 3$.
- ii. $\delta_1 = \frac{\min(\varepsilon_4, \varepsilon_2)}{(g+s)^2}$ where ε_4 is as in Proposition B,
- iii. $\delta = \delta_1 M'^{-1 - \frac{1}{2n}}$ where $M' = \max\left(2, \left\lceil \frac{K_2}{\delta_1} \right\rceil^2\right)^n$.

Since $\Gamma_\delta \subset (\Gamma_0^g)_\delta$,

$$S_2(C, \Gamma_\delta) \subset S_2(C, (\Gamma_0^g)_\delta).$$

In Lemma 4.5 with $\varepsilon = \delta$, we saw that

$$S_2(C, (\Gamma_0^g)_\delta) \subset \bigcup_{\substack{\phi \text{ Gauss-reduced} \\ \text{rk}\phi=2}} C(\overline{\mathbb{Q}}) \cap (B_\phi + (\Gamma_0^g)_\delta).$$

Note that $\delta < \delta_1 < \min(\varepsilon_1, \frac{K_1}{g})$. Then, Proposition 10.2 with $\varepsilon = \delta$ implies that

$$\bigcup_{\substack{\phi \text{ Gauss-reduced} \\ \text{rk}\phi=2}} C(\overline{\mathbb{Q}}) \cap (B_\phi + (\Gamma_0^g)_\delta) \hookrightarrow \bigcup_{\substack{\tilde{\phi}=(N\phi|\phi') \text{ special} \\ \text{rk}\phi=2}} (C(\overline{\mathbb{Q}}) \times \gamma) \cap (B_{\tilde{\phi}} + \mathcal{O}_\delta).$$

Note that $\delta = \delta_1 M'^{-(1+\frac{1}{2n})}$ and $\delta_1 \leq \varepsilon_2$. Then, Proposition A ii. in section 12 below, with $\varepsilon = \delta_1$, $r = 2$ (and n is already defined as $2(g+s) - 4 + 1$), shows that

$$\bigcup_{\substack{\tilde{\phi} \text{ special} \\ \text{rk}\phi=2}} (C(\overline{\mathbb{Q}}) \times \gamma) \cap (B_{\tilde{\phi}} + \mathcal{O}_\delta)$$

is a subset of

$$(13) \quad \bigcup_{\substack{\tilde{\phi} \text{ special} \\ H(\tilde{\phi}) \leq M' \text{ rk}\phi=2}} (C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{(g+s)\delta_1/H(\tilde{\phi})^{1+\frac{1}{2n}}} \right).$$

Observe that in (13), $\tilde{\phi}$ ranges over finitely many morphisms, as $H(\tilde{\phi})$ is bounded by M' .

We have chosen $\delta_1 \leq \frac{\varepsilon_4}{(g+s)^2}$. Proposition B ii. in section 13 below with $\varepsilon = (g+s)\delta_1$, implies that for all $\tilde{\phi} = (N\phi|\phi')$ special of rank 2, the set

$$(C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{(g+s)\delta_1/H(\tilde{\phi})^{1+\frac{1}{2n}}} \right)$$

is finite. Note that $H(\phi) \leq H(\tilde{\phi})$, thus also the sets

$$(C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{(g+s)\delta_1/H(\tilde{\phi})^{1+\frac{1}{2n}}} \right)$$

appearing in (13) are finite.

It follows that, the set $S_2(C, \Gamma_\delta)$ is contained in the union of finitely many finite sets. So it is finite. \square

Despite our proof relying on Dirichlet's Theorem and a Bogomolov type bound, a direct use of these two theorems is not sufficient to prove Theorem 1.5. Using Dirichlet's Theorem in a more natural way, one can prove that, for $r \geq 2$,

$$S_r(C, \Gamma_\varepsilon) \subset \bigcup_{H(\phi) \leq M(\varepsilon), \text{ rk}\phi=r} C(\overline{\mathbb{Q}}) \cap (B_\phi + \Gamma_\varepsilon).$$

On the other hand, a direct use of Bogomolov's type bound gives that

$$C(\overline{\mathbb{Q}}) \cap (B_\phi + \mathcal{O}_{\varepsilon/H(\phi)^2})$$

is finite, for ϕ of rank at least 2. Even if we forget Γ , the discrepancy between ε and $\varepsilon/H(\phi)^2$ does not look encouraging, and it took us a long struggle to overcome the problem. In Propositions A and B, we succeed in overcoming the mismatch; in both statements we obtain neighbourhoods of radius $\varepsilon/H(\phi)^{1+\frac{1}{2n}}$.

Do not be misled by the following wrong thought: One might think that, since we consider only morphisms ϕ such that $H(\phi) \leq M$, it could be enough to choose $\varepsilon' = \varepsilon/M^2$. However, $M = M(\varepsilon)$ is an unbounded function of ε as ε tends to 0.

12. PART I: THE BOX PRINCIPLE AND THE REDUCTION TO A FINITE SUB-UNION

In Lemma 12.2, we approximate a Gauss-reduced morphism with a Gauss-reduced morphism of bounded height. On a set of bounded height, such an approximation allows us to consider unions over finitely many algebraic subgroups, instead of unions over all algebraic subgroups (see Proposition A below).

We recall Dirichlet's Theorem on the rational approximation of reals.

Theorem 12.1 (Dirichlet 1842, see [14] Theorem 1 p. 24). *Suppose that $\alpha_1, \dots, \alpha_n$ are n real numbers and that $Q \geq 2$ is an integer. Then there exist integers f, f_1, \dots, f_n with*

$$1 \leq f < Q^n \quad \text{and} \quad |\alpha_i f - f_i| \leq \frac{1}{Q}$$

for $1 \leq i \leq n$.

Lemma 12.2. *Let $Q \geq 2$ be an integer. Let $\phi = (aI_r|L) \in M_{r \times g}(\mathbb{Z})$ be Gauss-reduced. Then, there exists a Gauss-reduced $\psi = (fI_r|L') \in M_{r \times g}(\mathbb{Z})$ such that*

- i. $H(\psi) = f \leq Q^{rg-r^2+1}$,
- ii. $\left| \frac{\psi}{f} - \frac{\phi}{a} \right| \leq Q^{-\frac{1}{2}} f^{-1 - \frac{1}{2(rg-r^2+1)}}.$

The norm $|\cdot|$ of a matrix is the maximum of the absolute values of its entries.

Proof. If $a \leq Q^{rg-r^2+1}$, no approximation is needed as ϕ itself satisfies the conclusion. So we can assume that

$$\phi = \begin{pmatrix} a & \dots & 0 & L_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & a & L_r \end{pmatrix}$$

is a Gauss-reduced morphism such that $H(\phi) = a > Q^{rg-r^2+1}$. Consider the element

$$\alpha = \left(1, \frac{L_1}{a}, \dots, \frac{L_r}{a} \right) = (\alpha_1, \alpha_2, \dots, \alpha_{rg-r^2+1})$$

in \mathbb{R}^{rg-r^2+1} .

Define $n = rg - r^2 + 1$. Apply Dirichlet's Theorem to α . Then, there exist integers f, f_1, \dots, f_n with

$$(14) \quad 1 \leq f < Q^n \quad \text{and} \quad |\alpha_i f - f_i| \leq \frac{1}{Q}$$

for $1 \leq i \leq n$. We can assume that f, f_1, \dots, f_n have greatest common divisor 1. Define

$$w = \frac{1}{f}(f_1, \dots, f_n) = \frac{1}{f}(f_1, L'_1, \dots, L'_r),$$

with $L'_i \in \mathbb{Z}^{g-r}$. We claim that

$$\begin{aligned} f_1 &= f, \\ |f_i| &\leq f. \end{aligned}$$

In fact, by (14) for $i = 1$ we have $\left| \frac{f_1}{f} - 1 \right| \leq \frac{1}{Qf}$, which implies that $|f - f_1| < 1$. Since f and f_1 are integers, we must have $f = f_1$. Similarly, by (14) for $i = 2, \dots, n$, we have $\left| \frac{f_i}{f} - \alpha_i \right| \leq \frac{1}{Qf}$. This implies that $|f_i| \leq f + \frac{1}{Q}$. We deduce $|f_i| \leq f$.

It follows that

$$\psi = \begin{pmatrix} f & \dots & 0 & L'_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & f & L'_r \end{pmatrix}$$

is a Gauss-reduced morphism of rank r with $H(\psi) = f$.

Relation (14) immediately gives

$$f \leq Q^n$$

and

$$\left| \frac{\psi}{f} - \frac{\phi}{a} \right| \leq \frac{1}{Qf} \leq \frac{1}{Q^{\frac{1}{2}} f^{1+\frac{1}{2n}}},$$

where in the last inequality we use $Q^{\frac{1}{2}} \geq f^{\frac{1}{2n}}$. □

At last we prove our first main proposition; the union can be taken over finitely many algebraic subgroups.

If ϕ has large height and B_ϕ is close to x , with x in a set of bounded height, then there exists ψ with height bounded by a constant such that B_ψ is also close to x . One shall be careful that, in the following inclusions, on the left hand side we consider a neighbourhood of B_ϕ of fixed radius, while on the right hand side the neighbourhood becomes smaller as the height of ψ grows. This is a crucial gain, with respect to the simpler approximation (obtained by a direct use of Dirichlet's Theorem) where the neighbourhoods have constant radius on both hand sides.

Proposition A. *Assume $r \geq 2$.*

i. *If $0 < \varepsilon \leq \varepsilon_1$, then*

$$\bigcup_{\phi} C(\overline{\mathbb{Q}}) \cap \left(B_\phi + (\Gamma_0^g)_{\varepsilon/M^{1+\frac{1}{2n}}} \right) \subset \bigcup_{H(\psi) \leq M} C(\overline{\mathbb{Q}}) \cap \left(B_\psi + (\Gamma_0^g)_{g\varepsilon/H(\psi)^{1+\frac{1}{2n}}} \right),$$

where ϕ and ψ range over Gauss-reduced morphisms of rank r , $n = rg - r^2 + 1$ and $M = \max(2, \lceil \frac{K_1}{\varepsilon} \rceil^2)^n$.

ii. *If $0 < \varepsilon \leq \varepsilon_2$, then*

$$\bigcup_{\tilde{\phi}} (C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/M'^{1+\frac{1}{2n}}} \right) \subset \bigcup_{H(\tilde{\psi}) \leq M'} (C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\psi}} + \mathcal{O}_{(g+s)\varepsilon/H(\tilde{\psi})^{1+\frac{1}{2n}}} \right),$$

where $\tilde{\phi}$ and $\tilde{\psi}$ range over special morphisms of rank r , $n = r(g+s) - r^2 + 1$ and $M' = \max(2, \lceil \frac{K_2}{\varepsilon} \rceil^2)^n$.

Proof. Part i. Let $\phi = (aI_r | L)$ be Gauss-reduced of rank r .

First consider the case $H(\phi) \leq M$. Then $\varepsilon/M^{1+\frac{1}{2n}} \leq \varepsilon/H(\phi)^{1+\frac{1}{2n}}$. Obviously

$$C(\overline{\mathbb{Q}}) \cap (B_\phi + \Gamma_0^g + \mathcal{O}_{\varepsilon/M^{1+\frac{1}{2n}}}) \subset C(\overline{\mathbb{Q}}) \cap (B_\phi + \Gamma_0^g + \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}})$$

is contained in the right hand side.

Secondly consider the case $H(\phi) > M$. We shall show that there exists ψ Gauss-reduced with $H(\psi) \leq M$ such that

$$C(\overline{\mathbb{Q}}) \cap \left(B_\phi + \Gamma_0^g + \mathcal{O}_{\varepsilon/M^{1+\frac{1}{2n}}} \right) \subset C(\overline{\mathbb{Q}}) \cap \left(B_\psi + \Gamma_0^g + \mathcal{O}_{g\varepsilon/H(\psi)^{1+\frac{1}{2n}}} \right).$$

We fix $Q = \max(2, \lceil \frac{K_1}{\varepsilon} \rceil^2)$. Recall that $n = rg - r^2 + 1$. By Lemma 12.2, there exists a Gauss-reduced morphism

$$\psi = \begin{pmatrix} f & \dots & 0 & L'_1 \\ \vdots & & \vdots & \\ 0 & \dots & f & L'_r \end{pmatrix}$$

such that

$$H(\psi) = f \leq M$$

and

$$(15) \quad \left| \frac{\psi}{f} - \frac{\phi}{a} \right| \leq \frac{1}{Q^{\frac{1}{2}} f^{1+\frac{1}{2n}}}.$$

Let $x \in C(\overline{\mathbb{Q}}) \cap (B_\phi + \Gamma_0^g + \mathcal{O}_{\varepsilon/M^{1+\frac{1}{2n}}})$. Then there exist $y \in \Gamma_0^g$ and $\xi \in \mathcal{O}_{\varepsilon/M^{1+\frac{1}{2n}}}$ such that

$$\phi(x - y - \xi) = 0.$$

We want to show that there exist $y' \in \Gamma_0^g$ and $\xi' \in \mathcal{O}_{g\varepsilon/f^{1+\frac{1}{2n}}}$ such that

$$\psi(x - y' - \xi') = 0.$$

Let y'' be a point such that

$$[a]y'' = \phi(y).$$

As Γ_0 is a division group, $y'' \in \Gamma_0^r$. We define

$$y' = (y'', 0) \in \Gamma_0^r \times \{0\}^{g-r},$$

then

$$\psi(y') = [f]y''.$$

Let ξ'' be a point such that

$$[f]\xi'' = \psi(x - y').$$

We define

$$\xi' = (\xi'', 0),$$

then

$$\psi(\xi') = [f]\xi'' = \psi(x - y').$$

So

$$\psi(x - y' - \xi') = 0.$$

It follows that

$$x \in C(\overline{\mathbb{Q}}) \cap (B_\psi + \Gamma_0^g + \mathcal{O}_{\|\xi'\|}).$$

In order to finish the proof, we are going to prove that

$$\|\xi'\| \leq \frac{g\varepsilon}{f^{1+\frac{1}{2n}}}.$$

By definition

$$\|\xi'\| = \|\xi''\| = \frac{\|\psi(x - y')\|}{f}.$$

Consider the equivalence

$$\begin{aligned} a\psi(x - y') &= a\psi(x) - a\psi(y') \\ &= a\psi(x) - a[f]y'' \\ &= a\psi(x) - f\phi(y) \\ &= a\psi(x) - f\phi(x) + f\phi(\xi). \end{aligned}$$

Then

$$\|\xi'\| = \frac{1}{af} \|f\phi(\xi) - f\phi(x) + a\psi(x)\| \leq \frac{1}{a} \|\phi(\xi)\| + \frac{1}{af} \|a\psi(x) - f\phi(x)\|.$$

Let us estimate separately each norm on the right.

On one hand

$$\frac{1}{a} \|\phi(\xi)\| \leq (g - r + 1) \|\xi\| \leq \frac{(g - 1)\varepsilon}{M^{1+\frac{1}{2n}}} \leq \frac{(g - 1)\varepsilon}{f^{1+\frac{1}{2n}}},$$

because $\|\xi\| \leq \varepsilon/M^{1+\frac{1}{2n}}$ and $f \leq M$.

On the other hand, since the rank of ϕ is at least 2 and $\varepsilon \leq \varepsilon_1$, we have that $x \in S_2(C, (\Gamma_0^g)_{\varepsilon_1})$, which has norm K_1 . Thus

$$\|x\| \leq K_1.$$

Using relation (15) and that $Q \geq \lceil \frac{K_1}{\varepsilon} \rceil^2$, it follows that

$$\begin{aligned} \frac{1}{af} \|a\psi(x) - f\phi(x)\| &\leq \left| \frac{\psi}{f} - \frac{\phi}{a} \right| \|x\| \\ &\leq \frac{1}{Q^{\frac{1}{2}} f^{1+\frac{1}{2n}}} \|x\| \\ &\leq \frac{\varepsilon \|x\|}{K_1 f^{1+\frac{1}{2n}}} \\ &\leq \frac{\varepsilon}{f^{1+\frac{1}{2n}}}. \end{aligned}$$

We conclude that

$$\|\xi'\| \leq \frac{(g-1)\varepsilon}{f^{1+\frac{1}{2n}}} + \frac{\varepsilon}{f^{1+\frac{1}{2n}}} \leq \frac{g\varepsilon}{f^{1+\frac{1}{2n}}}.$$

Part ii. We fix $Q = \max(2, \lceil \frac{K_2}{\varepsilon} \rceil^2)$.

Let $\tilde{\phi} = (N\phi|\phi') : E^{g+s} \rightarrow E^r$ be special. By the second definition of special

$$\tilde{\phi} = (bI_r|*)$$

is Gauss-reduced and $H(\tilde{\phi}) = b$.

As in part i.; if $H(\tilde{\phi}) \leq M'$ then $\varepsilon/M'^{1+\frac{1}{2n}} \leq \varepsilon/H(\tilde{\phi})^{1+\frac{1}{2n}}$ and

$$(C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/M'^{1+\frac{1}{2n}}} \right)$$

is contained in the right hand side.

Now, suppose that $H(\tilde{\phi}) > M'$. Recall that $n = r(g+s) - r^2 + 1$. By Lemma 12.2 (applied with $\phi = \tilde{\phi}$ and $\psi = \tilde{\psi}$) there exists $\tilde{\psi} = (fI_r|*)$ Gauss reduced such that

$$H(\tilde{\psi}) = f \leq M'$$

and

$$(16) \quad \left| \frac{\tilde{\phi}}{b} - \frac{\tilde{\psi}}{f} \right| \leq \frac{1}{Q^{\frac{1}{2}} f^{1+\frac{1}{2n}}}.$$

Then $\tilde{\psi}$ is special, according to the second formulation in Definition 10.1.

The proof is now similar to the proof of part i. We want to show that, if

$$\tilde{\phi}((x, \gamma) + \xi) = 0$$

for $\xi \in \mathcal{O}_{\varepsilon/M'^{1+\frac{1}{2n}}}$, then

$$\tilde{\psi}((x, \gamma) + \xi') = 0$$

for $\xi' \in \mathcal{O}_{(g+s)\varepsilon/H(\tilde{\psi})^{1+\frac{1}{2n}}}$.

Let ξ'' be a point in E^r such that

$$[f]\xi'' = -\tilde{\psi}(x, \gamma).$$

Let $\xi' = (\xi'', \{0\}^{g-r+s})$. Then

$$\tilde{\psi}((x, \gamma) + (\xi', 0)) = 0.$$

It follows

$$(x, \gamma) \in (C(\overline{\mathbb{Q}}) \times \gamma) \cap (B_{\tilde{\psi}} + \mathcal{O}_{\|\xi'\|}),$$

where $\tilde{\psi}$ is special and $H(\tilde{\psi}) \leq M'$.

It remains to prove that

$$\|\xi'\| \leq \frac{(g+s)\varepsilon}{H(\tilde{\psi})^{1+\frac{1}{2n}}}.$$

Obviously

$$b\tilde{\psi}(x, \gamma) = f \left(\tilde{\phi}(x, \gamma) - \tilde{\phi}(x, \gamma) \right) + b\tilde{\psi}(x, \gamma).$$

According to the definition of ξ' ,

$$\begin{aligned} \|\xi'\| = \|\xi''\| &= \frac{\|\tilde{\psi}(x, \gamma)\|}{f} = \frac{1}{bf} \left\| f \left(\tilde{\phi}(x, \gamma) - \tilde{\phi}(x, \gamma) \right) + b\tilde{\psi}(x, \gamma) \right\| \\ &\leq \frac{1}{b} \left\| \tilde{\phi}(x, \gamma) \right\| + \frac{1}{bf} \left\| b\tilde{\psi}(x, \gamma) - f\tilde{\phi}(x, \gamma) \right\|. \end{aligned}$$

We estimate the two norms on the right.

On one hand

$$\begin{aligned} \frac{\|\tilde{\phi}(x, \gamma)\|}{b} &= \frac{\|\tilde{\phi}(\xi)\|}{b} \leq (g-r+1+s)\|\xi\| \\ &\leq \frac{(g-r+1+s)\varepsilon}{M'^{1+\frac{1}{2n}}} \\ &\leq \frac{(g-r+1+s)\varepsilon}{f^{1+\frac{1}{2n}}}, \end{aligned}$$

where in the last inequality we use that $f \leq M'$.

On the other hand, by definition of ε_2 , we know that the norm of the set $S_2(C \times \gamma, \mathcal{O}_{\varepsilon_2})$ is bounded by K_2 . Since $\varepsilon \leq \varepsilon_2$, we have $(x, \gamma) \in S_2(C \times \gamma, \mathcal{O}_{\varepsilon_2})$. Therefore

$$\|(x, \gamma)\| \leq K_2.$$

Using relation (16) and that $Q \geq \lceil \frac{K_2}{\varepsilon} \rceil^2$, we estimate

$$\begin{aligned} \frac{1}{bf} \left\| b\tilde{\psi}(x, \gamma) - f\tilde{\phi}(x, \gamma) \right\| &\leq \left\| \frac{\tilde{\phi}}{b} - \frac{\tilde{\psi}}{f} \right\| \|(x, \gamma)\| \\ &\leq \frac{\|(x, \gamma)\|}{Q^{\frac{1}{2}} f^{1+\frac{1}{2n}}} \\ &\leq \frac{\varepsilon \|(x, \gamma)\|}{(K_2) f^{1+\frac{1}{2n}}} \leq \frac{\varepsilon}{f^{1+\frac{1}{2n}}}. \end{aligned}$$

Since $r \geq 2$, we conclude

$$\|\xi'\| \leq \frac{(g-1+s)\varepsilon}{f^{1+\frac{1}{2n}}} + \frac{\varepsilon}{f^{1+\frac{1}{2n}}} = \frac{(g+s)\varepsilon}{H(\tilde{\psi})^{1+\frac{1}{2n}}}.$$

□

13. PART II: THE ESSENTIAL MINIMUM AND THE FINITENESS OF EACH INTERSECTION

Up until now we have used, several times, the fact that the height of our sets is bounded (Theorem 1.2). In this section we often use that we work with a curve.

In the following, we set

$$n = 2(g+s) - 3.$$

We would like to use Conjecture 1.4 in order to provide $\varepsilon > 0$ such that, for all ϕ Gauss-reduced of rank $r = 2$, the set

$$(17) \quad (C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_\phi + \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right)$$

is finite. This set is simply

$$\phi_{|C \times \gamma}^{-1} \left(\phi(C \times \gamma) \cap \phi \left(\mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right) \right).$$

Further

$$\phi \left(\mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right) \subset \mathcal{O}_{g\varepsilon/H(\phi)^{\frac{1}{2n}}},$$

because if $\zeta \in \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}}$ then $\|\phi(\zeta)\| \leq gH(\phi)\|\zeta\| \leq g\varepsilon H(\phi)^{-\frac{1}{2n}}$. Thus, the set (17) is contained in the preimage of

$$\phi(C \times \gamma) \cap \mathcal{O}_{g\varepsilon/H(\phi)^{\frac{1}{2n}}}.$$

If we can ensure that there exists $\varepsilon > 0$ such that, for all morphisms ϕ Gauss-reduced of rank $r = 2$,

$$(18) \quad g\varepsilon H(\phi)^{-\frac{1}{2n}} < \mu(\phi(C \times \gamma)),$$

then the set (17) is finite.

It is noteworthy that a direct use of a Bogomolov type bound, even optimal, is not successful in the following sense: For a curve $X \subset E^g$ and any $\eta > 0$, Conjecture 1.4 provides an invariant $\epsilon(X, \eta)$ such that $\epsilon(X, \eta) < \mu(X)$. To ensure (18), we could naively require that

$$g\varepsilon H(\phi)^{-\frac{1}{2n}} \leq \epsilon(\phi(C \times \gamma), \eta)$$

for all ϕ Gauss-reduced of rank $r = 2$. Nevertheless this can be fulfilled only for $\varepsilon = 0$.

We need to throw new light on the problem to prove (18); via some isogenies, we construct a helping-curve D and then we relate its essential minimum to $C \times \gamma$. We then apply Conjecture 1.4 to D . Thus, we manage to provide a good lower bound for the essential minimum of $C \times \gamma$. We precisely take advantage of the fact that $\mu([b]C) = b\mu(C)$, while $\epsilon([b]C, \eta) = \frac{\epsilon(C, \eta)}{b^{\frac{1}{g-1}+2\eta}}$ for any positive integers b .

Let

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} a & 0 & L_1 \\ 0 & a & L_2 \end{pmatrix}$$

be a Gauss-reduced morphism of rank 2 with $H(\phi) = a$. We denote by $\bar{x} = (x_3, \dots, x_g)$, and recall that $n = 2(g + s) - 3$.

We define

$$a_0 = \lfloor a^{\frac{1}{2n}} \rfloor.$$

We associated to the morphism ϕ an isogeny

$$\Phi : E^g \rightarrow E^g$$

$$(x_1, \dots, x_g) \rightarrow (a_0\phi(x), x_3, \dots, x_g).$$

We then relate it to the isogenies:

$$A : E^g \rightarrow E^g$$

$$(x_1, \dots, x_g) \rightarrow (x_1, x_2, ax_3, \dots, ax_g).$$

$$A_0 : E^g \rightarrow E^g$$

$$(x_1, \dots, x_g) \rightarrow (x_1, x_2, a_0x_3, \dots, a_0x_g).$$

$$L : E^g \rightarrow E^g$$

$$(x_1, \dots, x_g) \rightarrow (x_1 + L_1(\bar{x}), x_2 + L_2(\bar{x}), x_3, \dots, x_g).$$

Definition 13.1 (Helping-curve). *We define the curve D to be an irreducible component of $A_0^{-1}LA^{-1}(C)$, where $(\cdot)^{-1}$ simply means the pre-image.*

The obvious relation

$$[a_0 a]D = \Phi(C)$$

is going to play a key role in the following.

We shall estimate degrees, as the Bogomolov type bound depends on the degree of the curve.

Lemma 13.2.

- i. *The degree of the curve $\phi(C)$ in E^2 is bounded by $6ga^2 \deg C$.*
- ii. *The degree of the curve D in E^g is bounded by $12g^2 a_0^{2(g-2)} a^{2(g-1)} \deg C$.*

Proof. i. Consider

$$\deg \phi(C) = \sum_{i=1}^2 \phi(C) \cdot H_i,$$

where H_i is the coordinate hyperplane given by $3x_i = 0$. The intersection number $\phi(C) \cdot H_i$ is bounded by the degree of the morphism $\phi_{i|C} : C \rightarrow E$. Recall that $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. By Bezout's Theorem, $\deg \phi_{i|C}$ is at most $3ga^2 \deg C$ - see [16] p. 61. Therefore

$$\deg \phi(C) \leq 6ga^2 \deg C.$$

ii. Let X be a generic transverse curve in E^g . By Hindry [10] Lemma 6 part i., we deduce

$$\begin{aligned} \deg A^{-1}(X) &\leq 2a^{2(g-2)} \deg X, \\ \deg A_0^{-1}(X) &\leq 2a_0^{2(g-2)} \deg X. \end{aligned}$$

To estimate $\deg L(X)$, we proceed as in part i.,

$$\deg L(X) = \sum_{i=1}^g L(X) \cdot H_i,$$

where H_i is given by $3x_i = 0$. The intersection number $L(X) \cdot H_i$ is bounded by the degree of the morphism $L'_{i|X} : X \rightarrow E$, where L'_i is the i th. row of L . By Bezout's Theorem $\deg L'_{i|X}$ is at most $3ga^2 \deg X$ - see [16] p. 61. Therefore

$$\deg L(X) \leq 3g^2 a^2 \deg X.$$

We conclude that

$$\begin{aligned} \deg D &\leq \deg A_0^{-1} L A^{-1}(C) \leq 2a_0^{2(g-2)} \deg L A^{-1}(C) \\ &\leq 6g^2 a_0^{2(g-2)} a^2 \deg A^{-1}(C) \\ &\leq 12g^2 a_0^{2(g-2)} a^2 a^{2(g-2)} \deg C. \end{aligned}$$

□

The following Proposition is a lower bound for the essential minimum of the image of a curve under Gauss-reduced morphisms. It reveals the dependence on the height of the morphism. While the first bound is an immediate application of Conjecture 1.4, the second estimate is subtle. Our lower bound for $\mu(\Phi(C + y))$ grows with $H(\phi)$. On the contrary, the Bogomolov type lower bound $\epsilon(\Phi(C + y))$ goes to zero like $(a_0 H(\phi))^{\frac{-1}{g-1} - \eta}$ - a nice gain.

Potentially, this suggests an interesting question; to investigate the behavior of the essential minimum under a general morphism.

Proposition 13.3. *Assume Conjecture 1.4. Then, for any point $y \in E^g(\overline{\mathbb{Q}})$ and any $\eta > 0$,*

i.

$$\mu(\phi(C + y)) > \epsilon_1(C, \eta) \frac{1}{a^{1+2\eta}},$$

where $\epsilon_1(C, \eta)$ is an effective constant depending on C and η . Recall that $a = H(\phi)$.

ii.

$$\mu(\Phi(C + y)) > \epsilon_2(C, \eta) a_0^{\frac{1}{g-1} - 8(g+s)(g-1)\eta},$$

where $\epsilon_2(C, \eta)$ is an effective constant depending on C , g and η . Recall that $a_0 = \lfloor a^{\frac{1}{2n}} \rfloor$.

Proof. Let us recall the Bogomolov type bound given in Conjecture 1.4; for a transverse irreducible curve X in E^g over $\overline{\mathbb{Q}}$ and any $\eta > 0$,

$$\epsilon(X, \eta) = \frac{c(g, E, \eta)}{\deg X^{\frac{1}{2\text{cod} X} + \eta}} < \mu(X).$$

i. Observe that $\phi(C) \subset E^2$ has codimension 1.

Let $q' = \phi(y)$. So $\phi(C + y) = \phi(C) + q'$. Since C is irreducible, transverse and defined over $\overline{\mathbb{Q}}$, $\phi(C) + q'$ is as well. Conjecture 1.4 gives

$$\mu(\phi(C + y)) = \mu(\phi(C) + q') > \epsilon(\phi(C) + q', \eta) = \frac{c(2, E, \eta)}{(\deg(\phi(C) + q'))^{\frac{1}{2} + \eta}}.$$

Degrees are preserved by translations, hence Lemma 13.2 i. implies that

$$\deg(\phi(C) + q') = \deg \phi(C) \leq 9ga^2 \deg C.$$

It follows that

$$\epsilon(\phi(C) + q', \eta) \geq \frac{c(2, E, \eta)}{(9ga^2 \deg C)^{\frac{1}{2} + \eta}}.$$

Define

$$\epsilon_1(C, \eta) = \frac{c(2, E, \eta)}{(9g \deg C)^{\frac{1}{2} + \eta}}.$$

Then

$$\mu(\phi(C + y)) \geq \frac{\epsilon_1(C, \eta)}{a^{1+2\eta}}.$$

ii. Let $q \in E^g$ be a point such that $[a_0 a]q = \Phi(y)$. Then

$$\Phi(C + y) = [a_0 a] (A_0^{-1} L A^{-1}(C) + q) = [a_0 a](D + q).$$

Therefore

$$(19) \quad \mu(\Phi(C + y)) = (a_0 a) \mu(D + q).$$

We now estimate $\mu(D + q)$ using Conjecture 1.4. The curve $D + q$ is irreducible by the definition of D . Since C is transverse and defined over $\overline{\mathbb{Q}}$, $D + q$ is also. Thus

$$\mu(D + q) > \epsilon(D + q, \eta) = \frac{c(g, E, \eta)}{\deg(D + q)^{\frac{1}{2(g-1)} + \eta}}.$$

Translations by a point preserves degrees, thus Lemma 13.2 ii. gives

$$\deg(D + q) = \deg D \leq 12g^2 a_0^{2(g-2)} a^{2(g-1)} \deg C.$$

Then

$$\epsilon(D + q, \eta) \geq \frac{c(g, E, \eta)}{(12g^2 \deg C)^{\frac{1}{2(g-1)} + \eta}} \left(a_0^{2(g-2)} a^{2(g-1)} \right)^{-\frac{1}{2(g-1)} - \eta}.$$

Define

$$\epsilon_2(C, \eta) = \frac{c(g, E, \eta)}{(12g^2 \deg C)^{\frac{1}{2(g-1)} + \eta}}.$$

So

$$\mu(D + q) \geq \epsilon_2(C, \eta) a_0^{-1 + \frac{1}{g-1} - 2(g-2)\eta} a^{-1-2(g-1)\eta}.$$

Substitute into (19), to obtain

$$\mu(\Phi(C + y)) > \epsilon_2(C, \eta) a_0^{\frac{1}{g-1} - 2(g-2)\eta} a^{-2(g-1)\eta}.$$

Recall that a_0 is the integral part of $a^{\frac{1}{2n}}$, where $n = 2(g + s) - 3$. So $2a_0 \geq a^{\frac{1}{2n}}$ and

$$a^{2(g-1)\eta} \leq (2a_0)^{4n(g-1)\eta}.$$

Further $2(g-2) + 4n(g-1) \leq 8(g+s)(g-1)$, so

$$\mu(\Phi(C + y)) > \epsilon_2(C, \eta) a_0^{\frac{1}{g-1} - 8(g+s)(g-1)\eta}.$$

□

Thankfully we come to our second main proposition; each set in the union is finite. The proof of i. case (1) is delicate. In general $\mu(\pi(C)) \leq \mu(C)$, for π a projection on some factors. We shall rather find a kind of reverse inequality. On a set of bounded height this will be possible.

Proposition B. *Assume Conjecture 1.4. Then, there exists $\varepsilon_4 > 0$ such that*

- i. *For $\varepsilon \leq \varepsilon_4$, for all $y \in \Gamma_0^2 \times \{0\}^{g-2}$ and for all Gauss-reduced morphisms ϕ of rank 2, the set*

$$(C(\overline{\mathbb{Q}}) + y) \cap \left(B_\phi + \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right)$$

is finite.

- ii. *For $\varepsilon \leq \frac{\varepsilon_4}{g+s}$ and for all special morphisms $\tilde{\phi} = (N\phi|\phi')$ of rank 2, the set*

$$(C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right)$$

is finite.

Recall that $n = 2(g + s) - 3$.

Proof. **Part i.** Choose

$$\eta \leq \eta_0 = \frac{1}{2^4(g+s)(g-1)^2}.$$

Define

$$m = \max \left(2, \left(\frac{K_1}{\epsilon_2(C, \eta)} \right)^{\frac{g-1}{1-8(g+s)(g-1)^2\eta}} \right),$$

and choose

$$\varepsilon \leq \min \left(\varepsilon_1, \frac{K_1}{g}, \frac{\epsilon_1(C, \eta)}{gm^{4n}} \right),$$

where $\epsilon_1(C, \eta)$ and $\epsilon_2(C, \eta)$ are as in Proposition 13.3.

Recall that $H(\phi) = a$. We distinguish two cases:

- (1) $a_0 = \lfloor a^{\frac{1}{2n}} \rfloor \geq m$,
- (2) $a_0 = \lfloor a^{\frac{1}{2n}} \rfloor \leq m$.

Case (1) - $\boxed{a_0 \geq m}$

Let $x + y \in (C(\overline{\mathbb{Q}}) + y) \cap (B_\phi + \mathcal{O}_{\varepsilon/a^{1+\frac{1}{2n}}})$, where

$$y = (y_1, y_2, 0, \dots, 0) \in \Gamma_0^2 \times \{0\}^{g-2}.$$

Then

$$\phi(x + y) = \phi(\xi)$$

for $\|\xi\| \leq \varepsilon/a^{1+\frac{1}{2n}}$ and $y \in \Gamma_0^g$.

We have chosen $\varepsilon \leq \varepsilon_1$, so $x \in S_2(C, (\Gamma_0^g)_{\varepsilon_1})$ which is a set of norm K_1 . Then

$$\|x\| \leq K_1.$$

Recall that $\Phi(z_1, \dots, z_g) = (a_0\phi(z), z_3, \dots, z_g)$. So

$$\begin{aligned} \Phi(x+y) &= (a_0\phi(x+y), x_3, \dots, x_g) \\ &= (a_0\phi(\xi), x_3, \dots, x_g). \end{aligned}$$

Therefore

$$\|\Phi(x+y)\| = \|(a_0\phi(\xi), x_3, \dots, x_g)\| \leq \max(a_0\|\phi(\xi)\|, \|x\|).$$

Since $\|\xi\| \leq \varepsilon a^{-(1+\frac{1}{2n})}$, $a_0 \leq a^{\frac{1}{2n}}$ and $\varepsilon \leq \frac{K_1}{g}$, we have that

$$a_0\|\phi(\xi)\| \leq a_0(g-r+1)\frac{\varepsilon}{a^{\frac{1}{2n}}} \leq K_1.$$

Also $\|x\| \leq K_1$. Thus

$$\|\Phi(x+y)\| \leq K_1.$$

We work under the hypothesis $a_0 \geq m = \left(\frac{K_1}{\epsilon_2(C, \eta)}\right)^{\frac{g-1}{1-8(g+s)(g-1)^2\eta}}$. So

$$K_1 \leq \epsilon_2(C, \eta) a_0^{\frac{1}{g-1}-8(g+s)(g-1)\eta}.$$

In Proposition 13.3 ii., we have proven that

$$\epsilon_2(C, \eta) a_0^{\frac{1}{g-1}-8(g+s)(g-1)\eta} < \mu(\Phi(C+y)).$$

So

$$\|\Phi(x+y)\| \leq K_1 < \mu(\Phi(C+y)).$$

We deduce that $\Phi(x+y)$ belongs to the finite set

$$\Phi(C+y) \cap \mathcal{O}_{K_1}.$$

The morphism $C+y \rightarrow \Phi(C+y)$ has finite fiber. We can conclude that since $\varepsilon \leq \min(\varepsilon_1, \frac{K_1}{g})$, for every ϕ Gauss-reduced of rank 2 with $a_0 = \lfloor a^{\frac{1}{2n}} \rfloor \geq m$, the set

$$(C(\overline{\mathbb{Q}}) + y) \cap \left(B_\phi + \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}}\right)$$

is finite.

Case (2) - $\boxed{a_0 \leq m}$

Let $x+y \in (C(\overline{\mathbb{Q}}) + y) \cap (B_\phi + \mathcal{O}_{\varepsilon/a^{1+\frac{1}{2n}}})$, where $y \in \Gamma_0^2 \times \{0\}^{g-2}$. Then

$$\phi(x+y) = \phi(\xi)$$

for $\|\xi\| \leq \varepsilon/a^{1+\frac{1}{2n}}$. However we have chosen $\varepsilon \leq \epsilon_1(C, \eta)/gm^{4n}$. Hence

$$\|\phi(x+y)\| = \|\phi(\xi)\| \leq \frac{g\varepsilon}{a^{\frac{1}{2n}}} \leq \frac{\epsilon_1(C, \eta)}{m^{4n}a^{\frac{1}{2n}}}.$$

We are working under the hypothesis $a_0 = \lfloor a^{\frac{1}{2n}} \rfloor \leq m$ and $m \geq 2$, so $a < (2a_0)^{2n} \leq m^{4n}$. Furthermore, $\eta \leq \eta_0 < \frac{1}{4n}$ implies that $a^{2\eta} < a^{\frac{1}{2n}}$. Thus

$$a^{1+2\eta} < m^{4n}a^{\frac{1}{2n}}.$$

And consequently

$$\|\phi(x+y)\| \leq \frac{\epsilon_1(C, \eta)}{m^{4n}a^{\frac{1}{2n}}} < \frac{\epsilon_1(C, \eta)}{a^{1+2\eta}}.$$

In Proposition 13.3 i. we have proven

$$\frac{\epsilon_1(C, \eta)}{a^{1+2\eta}} < \mu(\phi(C+y)).$$

We deduce that $\phi(x+y)$ belongs to the finite set

$$\phi(C+y) \cap \mathcal{O}_{\epsilon_1(C,\eta)m^{-4n}a^{-\frac{1}{2n}}}.$$

The morphism $C+y \rightarrow \phi(C+y)$ has finite fiber. We conclude that since $\varepsilon \leq \frac{\epsilon_1(C,\eta)}{gm^{4n}}$, for all ϕ Gauss-reduced of rank 2 with $a_0 = \lfloor a^{\frac{1}{2n}} \rfloor \leq m$, the set

$$(C(\overline{\mathbb{Q}}) + y) \cap \left(B_\phi + \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right)$$

is finite.

For the curve C , define

$$\epsilon(C) = \min(\epsilon_1(C, \eta_0), \epsilon_2(C, \eta_0)).$$

Note that $\left(\frac{\epsilon(C)}{gK_1} \right)^{8(g+s)g} \leq \frac{\epsilon_1(C,\eta)}{gm^{4n}}$. Thus, we could for instance choose

$$\varepsilon_4 = \min \left(\varepsilon_1, \frac{K_1}{g}, \left(\frac{\epsilon(C)}{gK_1} \right)^{8(g+s)g} \right).$$

Part ii. We want to show that, for every $\tilde{\phi} = (N\phi|\phi')$ special of rank 2, there exists ϕ Gauss-reduced of rank 2 and $y \in \Gamma_0^2 \times \{0\}^{g-2}$ such that the map $(x, \gamma) \rightarrow x+y$ defines an injection

$$(20) \quad (C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right) \hookrightarrow (C(\overline{\mathbb{Q}}) + y) \cap \left(B_\phi + \mathcal{O}_{(g+s)\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right).$$

We then apply part i. of this proposition; if $(g+s)\varepsilon \leq \varepsilon_4$, then

$$(C(\overline{\mathbb{Q}}) + y) \cap \left(B_\phi + \mathcal{O}_{(g+s)\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right)$$

is finite. So if $\varepsilon \leq \frac{\varepsilon_4}{g+s}$, then

$$(C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right)$$

is finite.

Let us prove the inclusion (20). Let $\tilde{\phi} = (N\phi|\phi')$ be special of rank 2. By definition of special $\phi = (aI_r|L)$ is Gauss-reduced of rank 2. Let

$$(x, \gamma) \in (C(\overline{\mathbb{Q}}) \times \gamma) \cap \left(B_{\tilde{\phi}} + \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right).$$

Then, there exists $\xi \in \mathcal{O}_{\varepsilon/H(\phi)^{1+\frac{1}{2n}}}$ such that

$$\tilde{\phi}((x, \gamma) + \xi) = 0.$$

Equivalently

$$N\phi(x) + \phi'(\gamma) + \tilde{\phi}(\xi) = 0.$$

Let $y' \in E^2$ be a point such that

$$N[a]y' = \phi'(\gamma).$$

Since Γ_0 is a division group,

$$y = (y', 0, \dots, 0) \in \Gamma_0^2 \times \{0\}^{g-2}$$

and

$$N\phi(y) = N[a]y' = \phi'(\gamma).$$

Therefore

$$N\phi(x+y) + \tilde{\phi}(\xi) = 0.$$

Let $\xi'' \in E^2$ be a point such that

$$N[a]\xi'' = \tilde{\phi}(\xi).$$

We define $\xi' = (\xi'', \{0\}^{g-2})$. Then

$$N\phi(\xi') = N[a]\xi'' = \tilde{\phi}(\xi),$$

and

$$N\phi(x + y + \xi') = 0.$$

Since $\tilde{\phi}$ is special $H(\tilde{\phi}) = Na$. Furthermore $\|\xi\| \leq \frac{\varepsilon}{a^{1+\frac{1}{2n}}}$. We deduce

$$\|\xi'\| = \|\xi''\| = \frac{\|\tilde{\phi}(\xi)\|}{Na} \leq \frac{(g+s)\varepsilon}{a^{1+\frac{1}{2n}}}.$$

In conclusion

$$N\phi(x + y + \xi') = 0$$

with $\|\xi'\| \leq \frac{(g+s)\varepsilon}{a^{1+\frac{1}{2n}}}$ and $y \in \Gamma_0^2 \times \{0\}^{g-2}$. Equivalently

$$(x + y) \in (C(\overline{\mathbb{Q}}) + y) \cap \left(B_{N\phi} + \mathcal{O}_{(g+s)\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right).$$

By Lemma 4.4 i. (with $\psi = \phi$), we deduce

$$(x + y) \in (C(\overline{\mathbb{Q}}) + y) \cap \left(B_{\phi} + \mathcal{O}_{(g+s)\varepsilon/H(\phi)^{1+\frac{1}{2n}}} \right),$$

with $y \in \Gamma_0^2 \times \{0\}^{g-2}$ and ϕ Gauss-reduced of rank 2.

This proves relation (20) and concludes the proof. \square

14. THE EFFECTIVENESS ASPECT

14.1. An effective weak height bound. We give an effective bound for the height of $S_1(C, \mathcal{O}_{\varepsilon})$ for C transverse.

Theorem 14.1. *Let C be transverse. For every real $\varepsilon \geq 0$, the norm of the set $S_1(C, \mathcal{O}_{\varepsilon})$ is bounded by $K_0 \max(1, \varepsilon)$, where K_0 is an effective constant depending on the degree and the height of C .*

Proof. If $x \in S_1(C, \mathcal{O}_{\varepsilon})$, there exist $\phi : E^g \rightarrow E$ and $\xi \in \mathcal{O}_{\varepsilon}$ such that $\phi(x - \xi) = 0$. Now the proof follows step by step the proof of [16] Theorem 1 page 55 where we replace \hat{h} by h , y by ϕ , p by x and $\hat{h}(y(p)) = 0$ by $h(\phi(x)) = c_0 \deg \phi h(\xi)$ with $h(\xi) \leq \varepsilon^2$. \square

14.2. The strong hypotheses and an effective weak theorem.

Proof of Theorem 1.6. The proof is similar to the proof of Theorem 1.5 given in section 11.

Theorem 14.1 implies that for $r \geq 1$ the norm of the set $S_r(C, \mathcal{O}_1)$ is bounded by an effective constant K_0 . Let

$$\begin{aligned} \text{i. } \delta_1 &= \frac{1}{g} \min \left(1, \frac{K_0}{g}, \left(\frac{\varepsilon(C)}{gK_0} \right)^{8g^2} \right). \\ \text{ii. } \delta &= \delta_1 M^{-(1+\frac{1}{2(2g-3)})} \quad \text{where } M = \max \left(2, \left\lceil \frac{K_0}{\delta_1} \right\rceil^2 \right)^{2g-3}. \end{aligned}$$

In section 12, Proposition A i. with $\Gamma = 0$, $\varepsilon_1 = 1$ and $K_1 = K_0$, we have shown that

$$\bigcup_{\substack{\phi \text{ Gauss reduced} \\ \text{rk}(\phi)=2}} C(\overline{\mathbb{Q}}) \cap (B_{\phi} + \mathcal{O}_{\delta}) \subset \bigcup_{\substack{\phi \text{ Gauss reduced} \\ \text{rk}(\phi)=2 \ H(\phi) \leq M}} C(\overline{\mathbb{Q}}) \cap \left(B_{\phi} + \mathcal{O}_{g\delta_1/H(\phi)^{1+\frac{1}{2(2g-3)}}} \right).$$

In section 13, Proposition B i. with $y = 0$, $s = 0$ and $n = 2g - 3$, $K_1 = K_0$ and relation (13), we have shown that for all ϕ Gauss-reduced of rank 2, the set

$$C(\overline{\mathbb{Q}}) \cap \left(B_\phi + \mathcal{O}_{g\delta_1/H(\phi)^{1+\frac{1}{2(2g-3)}}} \right)$$

is finite.

The union of finitely many finite sets is finite. It follows that

$$\bigcup_{\substack{\phi \text{ Gauss-reduced} \\ \text{rk}(\phi)=2}} C(\overline{\mathbb{Q}}) \cap (B_\phi + \mathcal{O}_\delta)$$

is finite.

By Lemma 4.5 i. we deduce that $S_2(C, \mathcal{O}_\delta)$ is finite.

This shows that Theorem 1.6 holds for

$$\varepsilon \leq \frac{1}{g^{4g}} \min(1, K_0^{-1})^{4g} \min \left(1, K_0, \left(\frac{\epsilon(C)}{gK_0} \right)^{8g^2} \right)^{4g}.$$

□

14.3. An effective bound for the cardinality of the sets. We have just shown that for C transverse, ε can be made effective. The purpose of this section is to indicate an effective bound for the cardinality of $S_2(C, \mathcal{O}_\varepsilon)$, under:

Conjecture 14.2 (S. David; personal communication). *Let C be a transverse curve in A . Then, there exist constants $c' = c'(g, \deg_L A, h_L(A), [k : \mathbb{Q}])$ and $c'' = c''(g, \deg_L A, h_L(A), [k : \mathbb{Q}])$ such that, for*

$$\begin{aligned} \epsilon(C) &= \frac{c'}{(\deg_L V)^{\frac{1}{2\text{cod } V}}} \\ \Theta(C) &= c''(\deg_L C)^g, \end{aligned}$$

the cardinality of $C(\overline{\mathbb{Q}}) \cap \mathcal{O}_{\epsilon(C)}$ is bounded by $\Theta(C)$.

This is the abelian analogue to [2] Conjecture 1.2.

We prove:

Theorem 14.3. *Let C be transverse. Assume that Conjecture 14.2 holds. Then, there exists an effective $\varepsilon > 0$ such that the cardinality of $S_2(C, \mathcal{O}_\varepsilon)$ is bounded by an effective constant.*

Proof. Let δ and δ_1 be as defined in the previous proof.

By Proposition A i. in section 12 we deduce that

$$S_2(C, \mathcal{O}_\delta) \subset \bigcup_{\substack{\phi \text{ Gauss-reduced} \\ H(\phi) \leq M}} C(\overline{\mathbb{Q}}) \cap \left(B_\phi + \mathcal{O}_{(g+1)\delta_1/H(\phi)^{1+\frac{1}{2(2g-3)}}} \right).$$

Note that, for any curve D and positive integers n , the cardinality of $[n]D \cap \mathcal{O}_{n\epsilon(D)}$ is still $\Theta(D)$. Going through the proofs of Proposition B i. in section 13, we see that

$$\#S_2(C, \mathcal{O}_\delta) \leq \sum_{H(\phi) \leq M} \# \left(\phi|_C^{-1} (\phi(C) \cap \mathcal{O}_{\epsilon(\phi(C))}) \right),$$

where $\phi|_C : C \rightarrow \phi(C)$ is the restriction of ϕ to C . Recall that the fiber of $\phi|_C$ has cardinality at most $3gH(\phi)^2 \leq 3gM^2$ (see [16] p. 61). We denote

$$\Delta_{\max} = \max_{H(\phi) \leq M} \#(\phi(C) \cap \mathcal{O}_{\epsilon(\phi(C))}).$$

We deduce

$$\#S_2(C, \mathcal{O}_\delta) \leq 3gM^3 \Delta_{\max}.$$

By Lemma 13.2 i., $\deg \phi(C) \leq (3gH(\phi))^2 \deg C$. Conjecture 14.2 implies that

$$\Delta_{\max} \leq (3gH(\phi))^{2g} \Theta(C)$$

with $\Theta(C)$ explicitly given. We deduce

$$(21) \quad \sharp S_2(C, \mathcal{O}_\delta) \leq (3g)^{2g+1} M^{2g+3} \Theta(C).$$

By Theorem 14.1 the constant K_0 is effective. So M is also effective. Thus the bound (21) is effective, for C transverse. \square

Similar computations imply a bound for the cardinality of $S_2(C, \Gamma_\delta)$.

For $\delta \leq \frac{\varepsilon_4}{(g+s)^2} M'^{-1-\frac{1}{4g+4s-6}}$, we obtain

$$\sharp S_2(C, \Gamma_\delta) \leq c_1(g) M'^{c_2(g,s)} \Theta(C).$$

Here $c_1(g)$ (and $c_2(g, s)$) are effective constants depending only on g (and s). M' depends explicitly on C , g and K_2 , while ε_4 depends explicitly on C , g , s and K_1 .

In view of Theorem 9.1, the above bound also implies a bound for the cardinality of $S_2(C \times \gamma, \mathcal{O}_{\delta/(g+s)K_4})$.

However, Theorem 1.2 does not give effective K_1 or K_2 . Consequently neither M' nor ε_4 are effective. An effective estimate for K_1 or K_2 would imply an effective Mordell Conjecture. This gives an indication of the difficulty to extend effective height proofs from transverse curves to weak-transverse curves.

15. FINAL REMARKS

15.1. The C.M. case. Sections 2 - 7 are proven for any E regardless of whether E has C.M. or not. Since Conjecture 1.4 is stated for any E , Proposition B holds unchanged for E with C.M.

We can extend Proposition A to Gauss-reduced $\phi \in M_{r,g}(\mathbb{Z} + \tau\mathbb{Z})$ as follows. Decompose $\phi = \phi_1 + \tau\phi_2$ for $\phi_i \in M_{r,g}(\mathbb{Z})$, then let the morphism $\psi = (\phi_1|\phi_2)$ act on $(x, \tau x) + (y, \tau y) + (\xi, \tau\xi)$ for $x \in S_r(C, (\Gamma_0^g)_\varepsilon)$, $y \in \Gamma_0^g$ and $\xi \in \mathcal{O}_\varepsilon$. Apply Proposition A to ψ . Constants will depend on τ .

15.2. From powers to products. In a power there are more algebraic subgroups than in a product where not all the factors are isogenous. If we consider a product of non-C.M. elliptic curves, then the matrix of a morphism ϕ is simply an integral matrix where the entries corresponding to non isogenous factors are zeros. So nothing changes with respect to our proofs. If the curve is in a product of elliptic curves in general, we shall extend the definition of Gauss-reduced, introducing constants $c_1(\tau)$ and $c_2(\tau)$, such that the element a on the diagonal has norm satisfying $c_1(\tau)H(\phi) \leq |a| \leq c_2(\tau)H(\phi)$.

We leave the details to the reader.

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